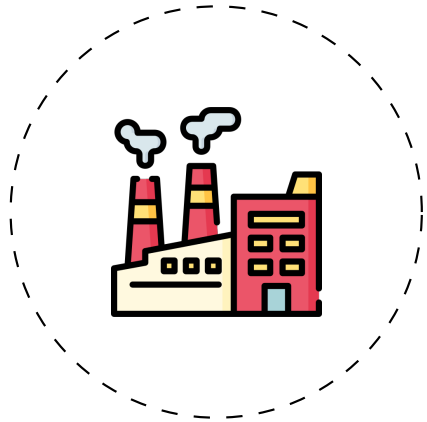
Growing a stochastic sub-tree for mixed-integer linear programming under uncertainty

Zoé Fornier, Bernardo Freitas Paulo Da Costa, Vincent Leclère, Mervé Bodur

ISMP - July 2024

MOTIVATION



$\mathcal{I}$ machines, $\mathcal{J}$ products
Process and Physical constraints

Example

Optimize the production plan of a factory with uncertain demand and energy prices.

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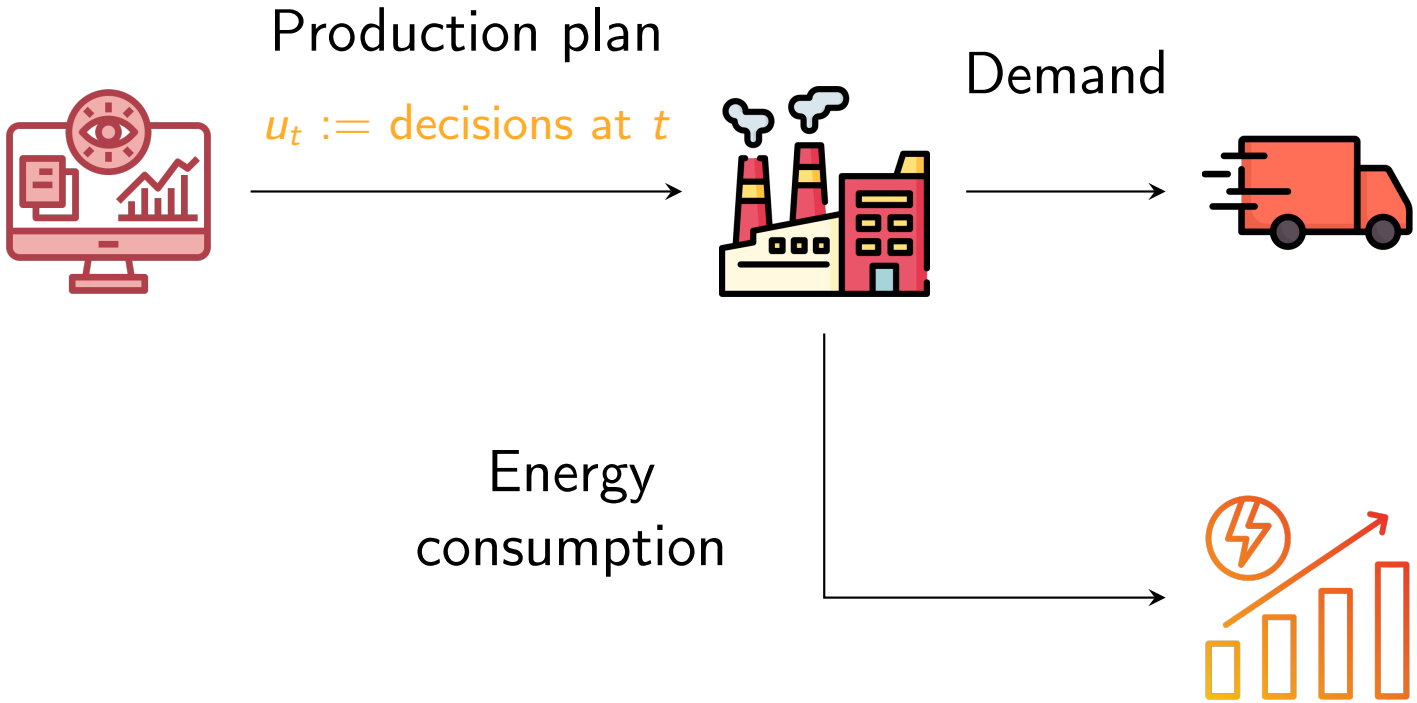
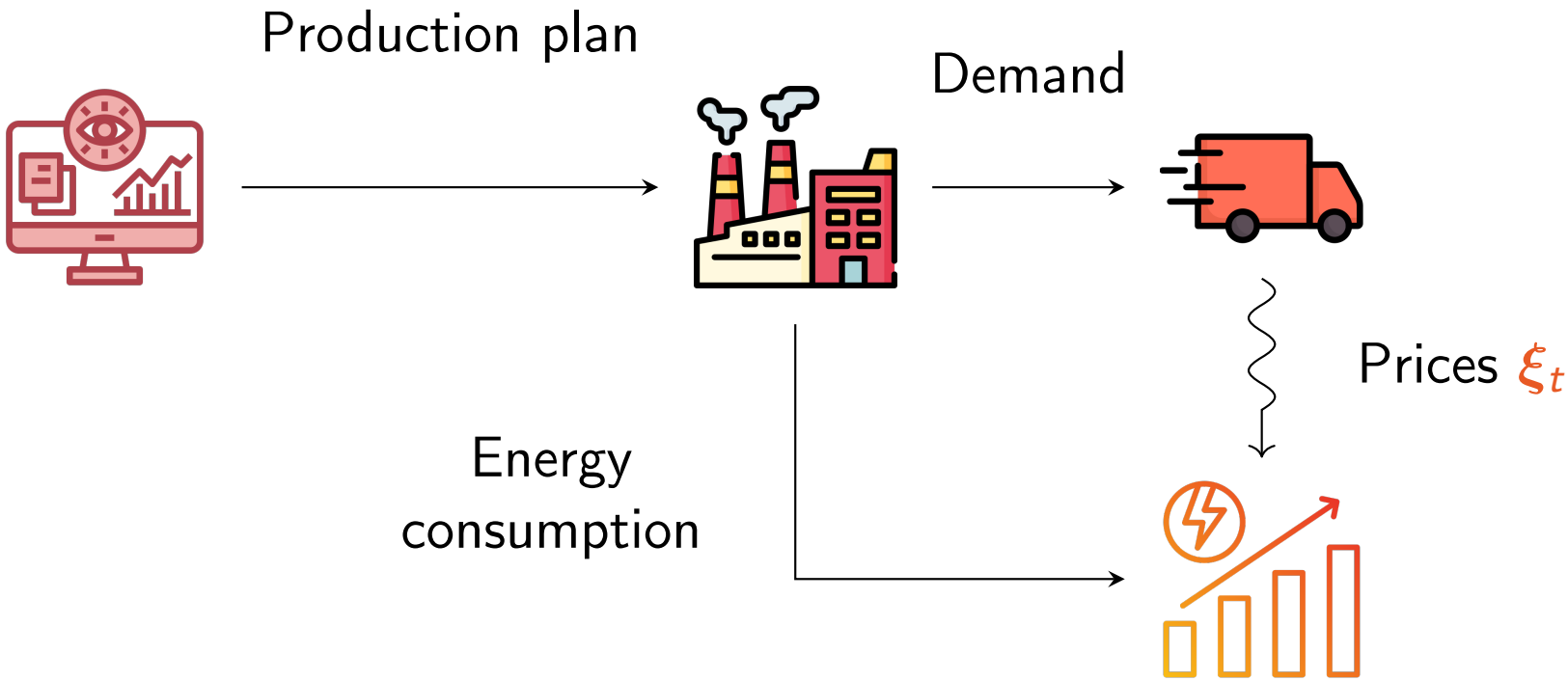


Example

Optimize the production plan of a factory with uncertain demand and energy prices.

- ➥ **Decision variables:** when/how much to produce
- ➥ **Costs:** energy purchases

MOTIVATION



Example

Optimize the production plan of a factory with uncertain demand and energy prices.

- **Randomness:** uncertain energy prices
- **Scale:** $T = 24$, $|\mathcal{I}| = 2$, $|\mathcal{J}| = 3$

SOLVING MiSLP

Multistage integer Stochastic Linear Problems

$$(P) \qquad \min_{\boldsymbol{x},\boldsymbol{u},\boldsymbol{b}}$$

$$\boldsymbol{x}_{t+1} = F_{t+1}(\boldsymbol{x}_t, \boldsymbol{u}_t, \boldsymbol{b}_t, \boldsymbol{\xi}_t) \qquad \forall t$$

- **State variables:** x_{t+1} follows the dynamic F_{t+1}

SOLVING MiSLP

Multistage integer Stochastic Linear Problems

$$
\begin{aligned}
(\mathrm{P}) \qquad \min_{\boldsymbol{x},\boldsymbol{u},\boldsymbol{b}} \quad & \\
& \boldsymbol{x_{t+1}} = F_{t+1}(\boldsymbol{x_t}, \boldsymbol{u_t}, \boldsymbol{b_t}, \boldsymbol{\xi_t}) && \forall t \\
& \boldsymbol{u_t} \in \mathcal{U}(\boldsymbol{x_t}, \boldsymbol{\xi_t}) \subset \mathbb{R}^{n_u} && \forall t \\
& \boldsymbol{b_t} \in \mathcal{B}(\boldsymbol{x_t}, \boldsymbol{\xi_t}) \subset \{0,1\}^{n_b} && \forall t
\end{aligned}
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- **Continuous Control** u_t continuous
- **Integer Control** b_t binary

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& \sigma(\boldsymbol{u_t}, \boldsymbol{b_t}) \subset \sigma(\boldsymbol{\xi_1}, \ldots, \boldsymbol{\xi_t}) && \forall t
\end{aligned}
$$

- **Randomness** $(\boldsymbol{\xi}_t)_{t\in[T]}$ is a sequence of finitely supported random variables
- **Non-anticipativity constraints:** we cannot know what will happen in the future

SOLVING MiSLP

Multistage integer Stochastic Linear Problems

$$
\begin{aligned}
(\mathrm{P}) \qquad \min_{\boldsymbol{x},\boldsymbol{u},\boldsymbol{b}} \quad & \mathbb{E}\left[\sum_{t=1}^{T} L_t(\boldsymbol{x_{t-1}}, \boldsymbol{u_t}, \boldsymbol{b_t}, \boldsymbol{\xi_t})\right] & \\
& \boldsymbol{x_{t+1}} = F_{t+1}(\boldsymbol{x_t}, \boldsymbol{u_t}, \boldsymbol{b_t}, \boldsymbol{\xi_t}) & \forall t \\
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\end{aligned}
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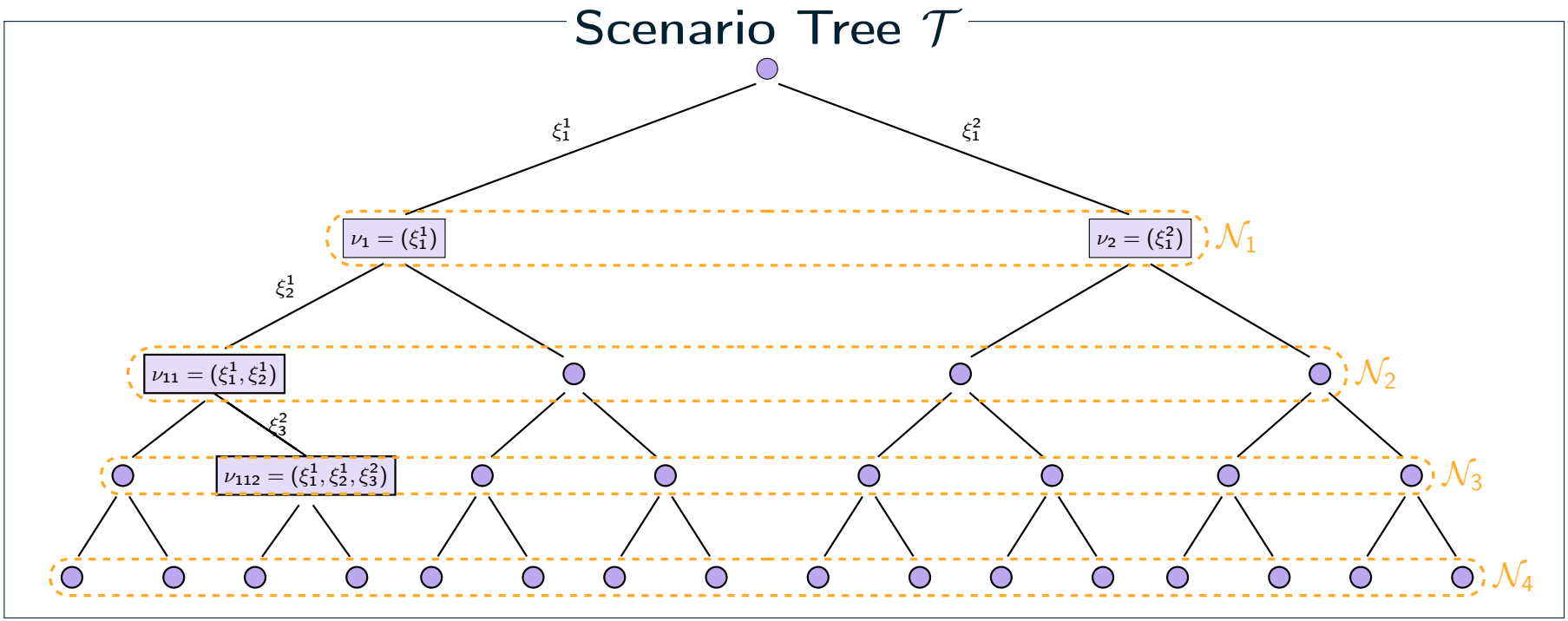
- **Objective** Minimize expected costs
- **Instantaneous cost:** L_t

SOME NOTATIONS



- A **scenario** $(\xi_t)_{t\in[T]}$ is a realization of $(\boldsymbol{\xi}_t)_{t\in[T]}$.
- The **scenario tree** $\mathcal{T}$ is the collection of all scenarios.
- $\mathcal{N}_t$ is the set of nodes in $\mathcal{T}$ of depth t.
- A **node** $\nu := (\xi_1, \xi_2, \ldots, \xi_\tau)$ reads all its ancestors.

REFORMULATION

We can always reformulate (P) as a large deterministic MILP:

MiSLP: extensive formulation

$$
\begin{aligned}
(\mathrm{P}_{\mathrm{ext}}) \qquad \min_{x_\nu, u_\nu, b_\nu} \; & \sum_{t=1}^{T} \sum_{\nu \in \mathcal{N}_t} \pi_\nu L_t(x_\nu, u_\nu, b_\nu, \xi_\nu) \\
& x_\nu = F_\nu(x_{a(\nu)}, u_{a(\nu)}, b_{a(\nu)}, \xi_{a(\nu)}) && \forall \nu \\
& u_\nu \in \mathcal{U}(x_\nu, \xi_\nu) && \forall \nu \\
& b_\nu \in \mathcal{B}(x_\nu, \xi_\nu) \subset \{0,1\}^{n_b} && \forall \nu
\end{aligned}
$$

- all variables are declined on each node ν
- the dynamics depend on the parent $a(\nu)$ of ν
- Intractable: if $\boldsymbol{\xi}_t$ is discretized with 10 values, $|\mathcal{T}| = 10^{24}$

PRESENTATION OUTLINE

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DYNAMIC PROGRAMMING PRINCIPLES

$V_\nu(x) :=$ optimal cost from node ν and state x.

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➥ with stagewise independence hypothesis, $V_\nu(x) = V_t(x)$

PRESENTATION OUTLINE

CURRENT NUMERICAL ALGORITHMS

- Stochastic Dynamic Programming (SDP)

  Principle: we solve the problem with dynamic equations, by discretizing continuous state variables.
  Pros: few assumptions, easily implemented.
  Cons: curse of dimensionality.

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- Stochastic Dual Dynamic integer programming (SDDiP)

  Principle: algorithm built on SDDP to solve multistage linear stochastic problems with only binary state variables.
  Pros: theoretical guarantees.
  Cons: slow iterations and convergence.

PRESENTATION OUTLINE

SDDP: FOCUS

Dynamic Programming: cost-to-go functions

$$V_t(x,\xi) = \min_{y,u} \quad L_t(x,u,\xi) + \theta \tag{2a}$$

$$y = F_t(x,u,\xi) \tag{2b}$$

$$u \in \mathcal{U}(y,\xi) \tag{2c}$$

$$\theta \geq f_{t+1,k} + g_{t+1,k}^T(y - x_{t+1,k}) \qquad \forall k \tag{2d}$$

Assumptions

- stage-wise independence of noises
- Continuous variables, V_t is a **convex** function of x
  - ➥ V_t can be approximated as a maximum of linear cuts

SDDP: ALGORITHM

Iteration k

$\boxed{\{V_t^{k-1}\}_{t\in[T]}}$

We dispose of
current approximation

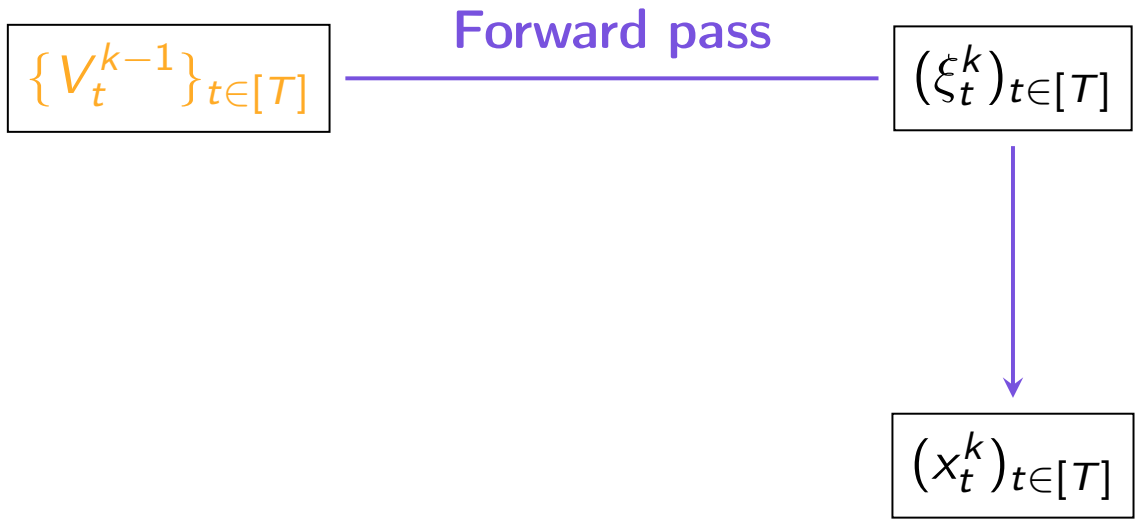
SDDP: ALGORITHM

Iteration k

We randomly draw a scenario

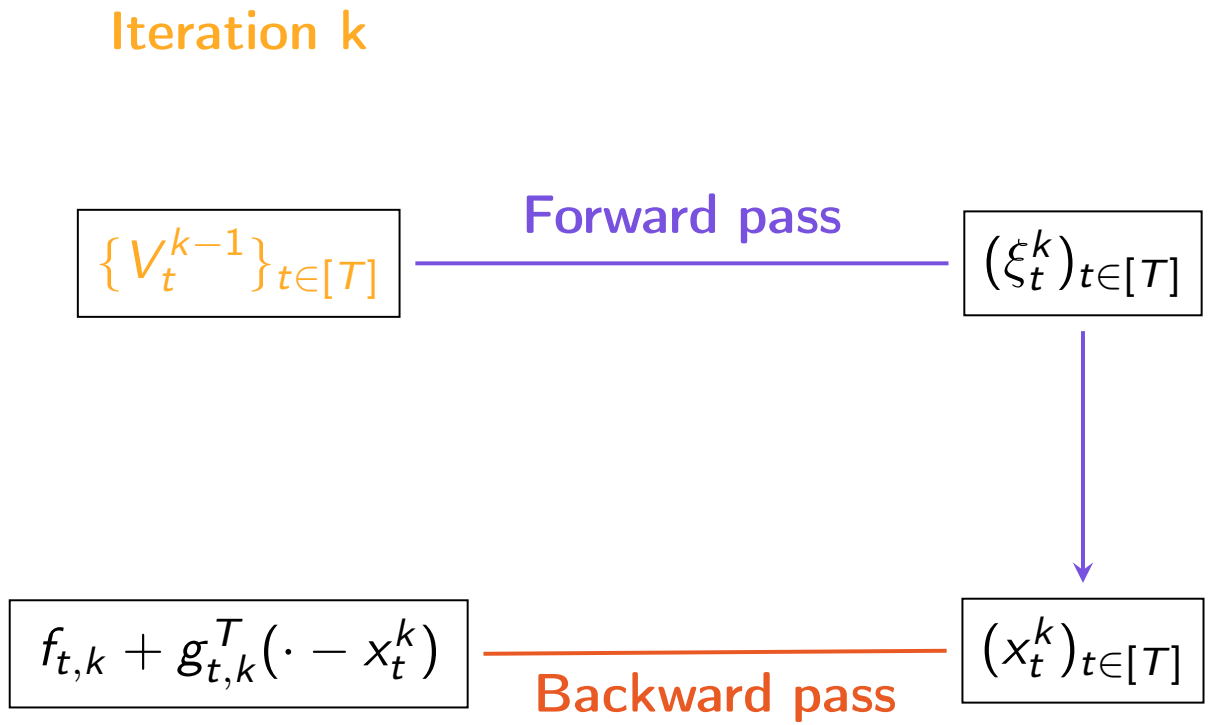
SDDP: ALGORITHM

Iteration k



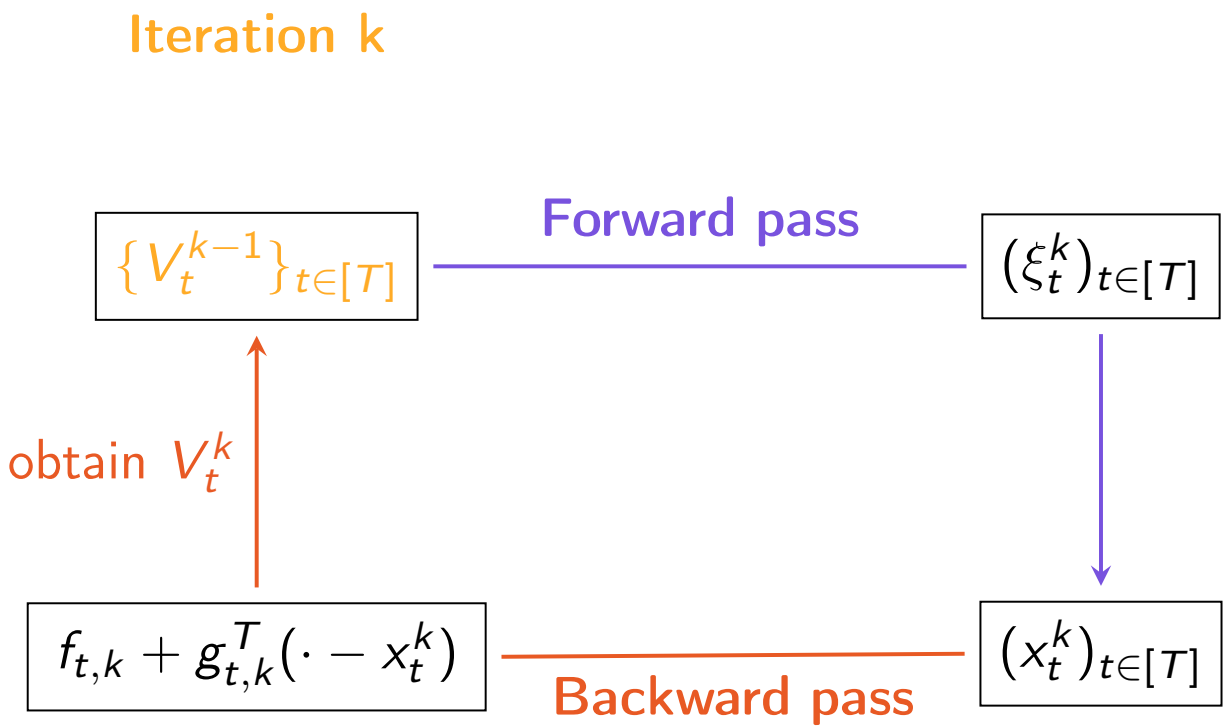
compute current optimal trajectory

SDDP: ALGORITHM



Compute new cuts to approximate of $\{V_t\}_{t\in[T]}$

SDDP: ALGORITHM



PRESENTATION OUTLINE

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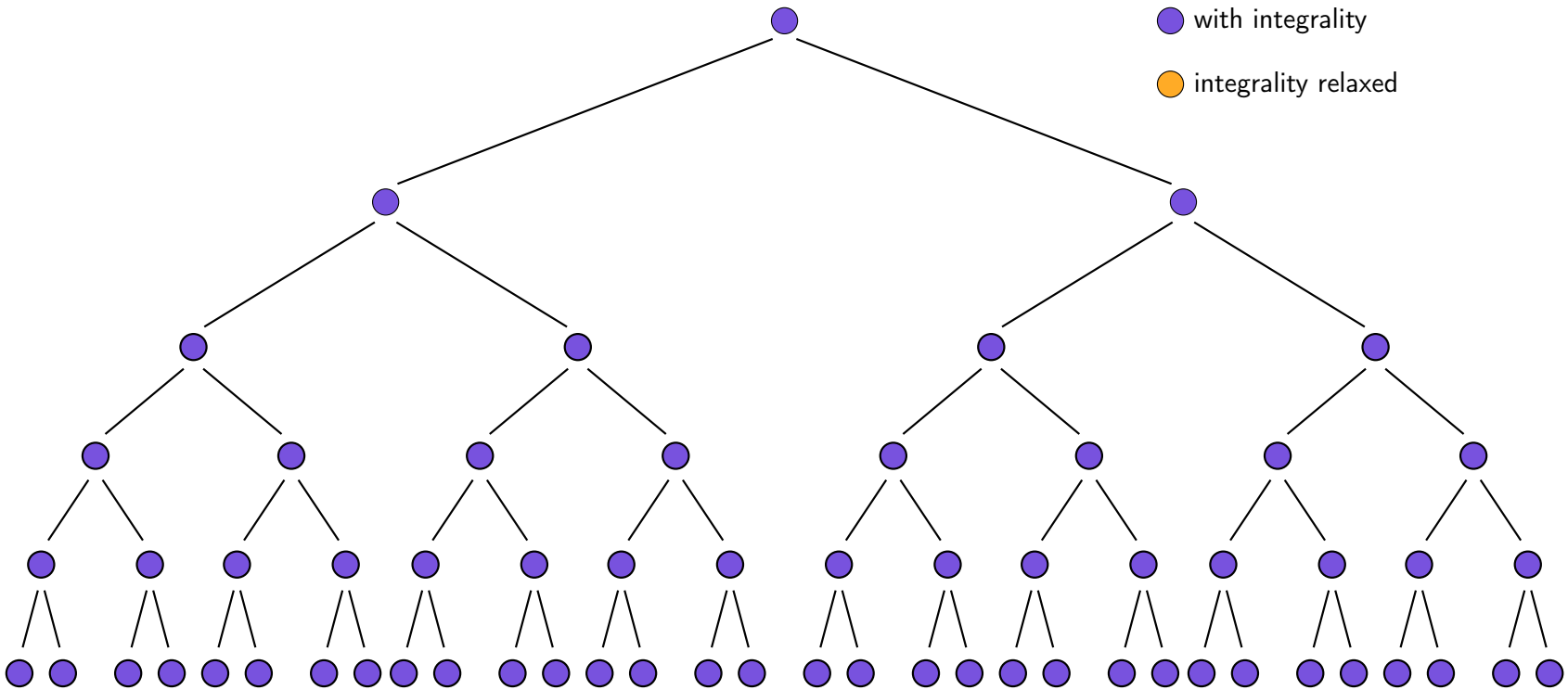
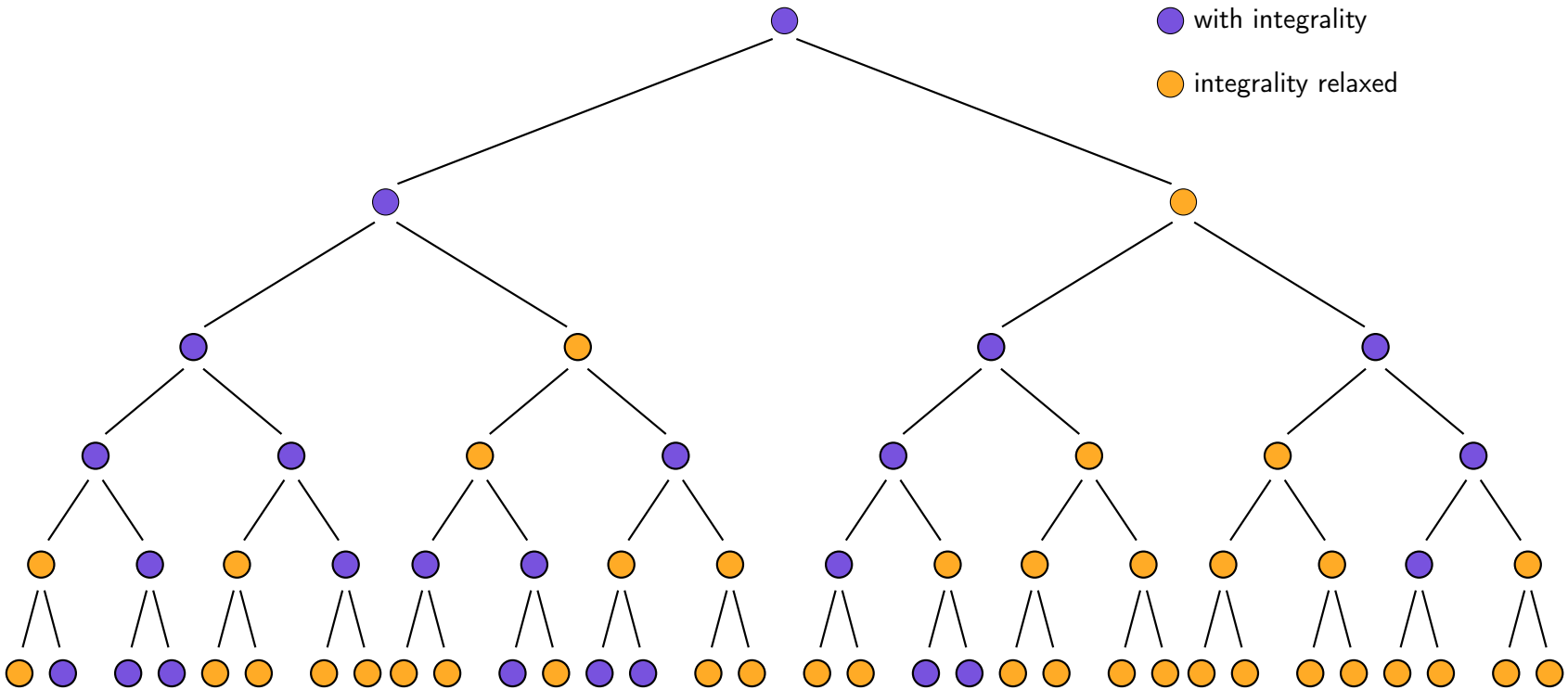
INTUITION

The full problem in its extensive formulation is intractable.



INTUITION

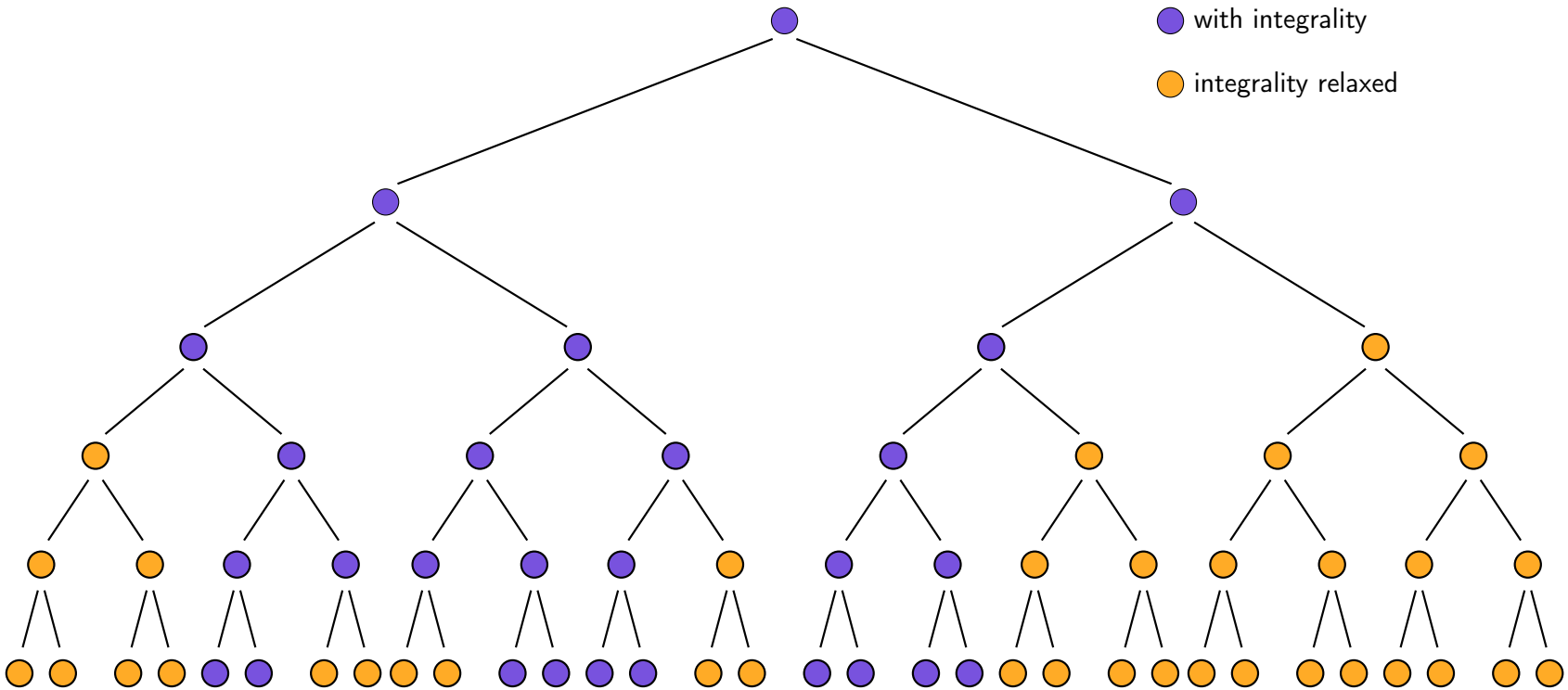
Idea: relax partially integrality.



➥ This problem is easier to solve, but still intractable.

INTUITION

Depending on how we relax intregrality, we can use SDDP.



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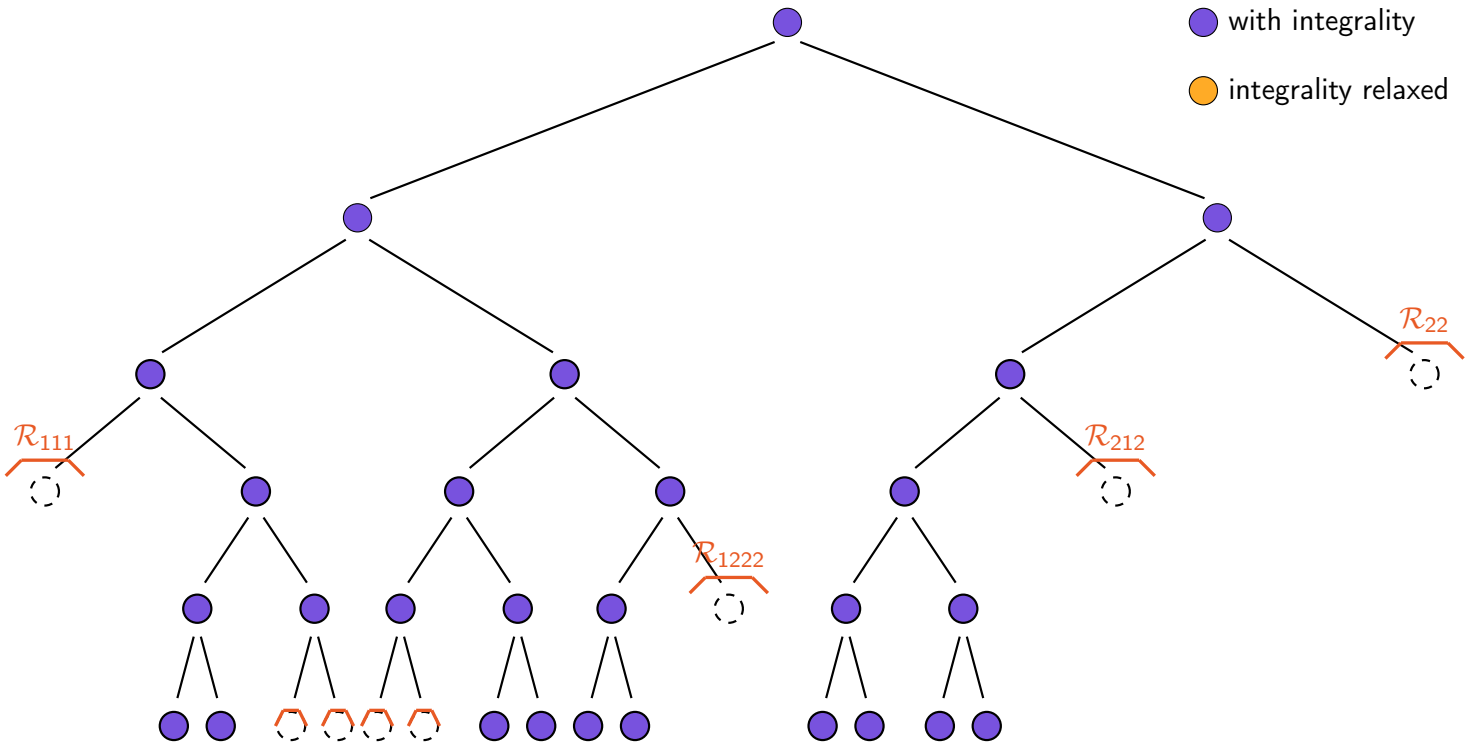

PRESENTATION OUTLINE

HOW TO GROW THE SUB-TREE?

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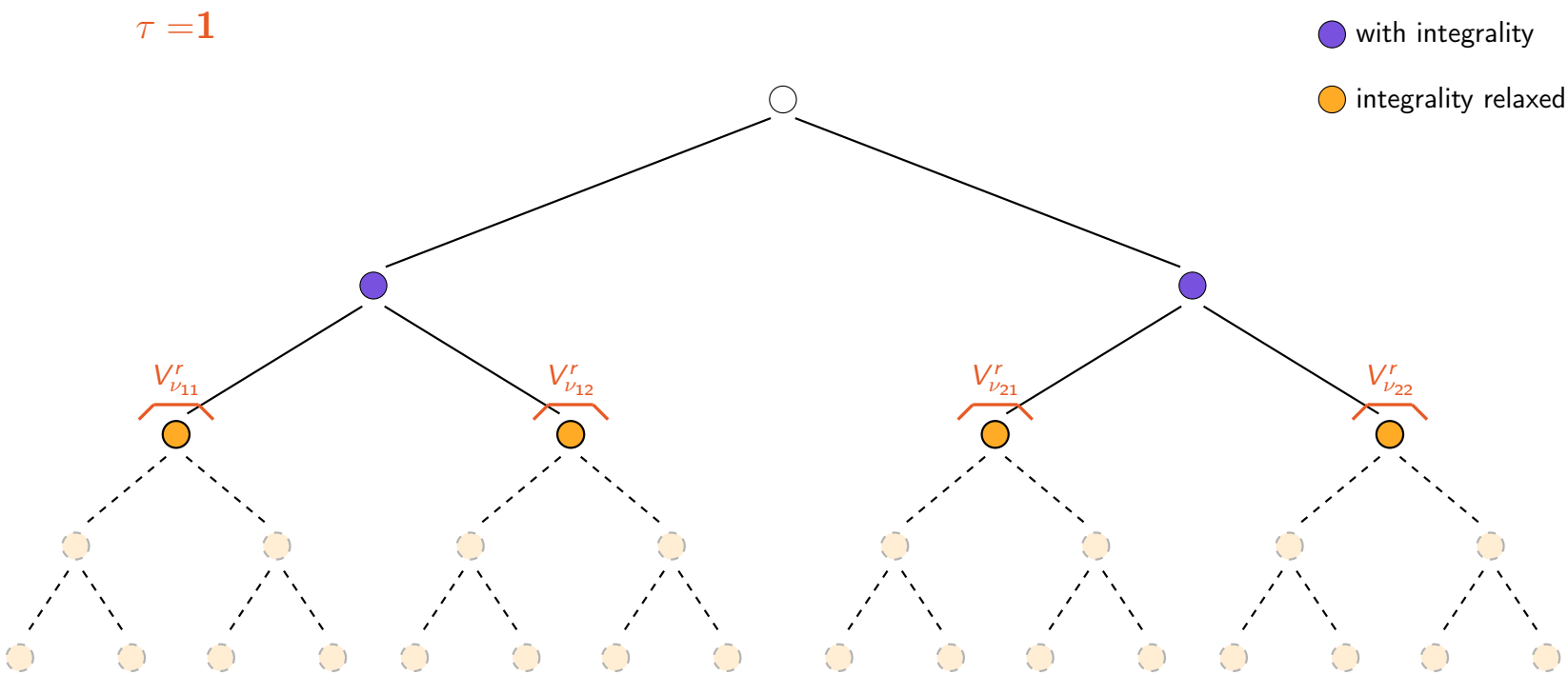

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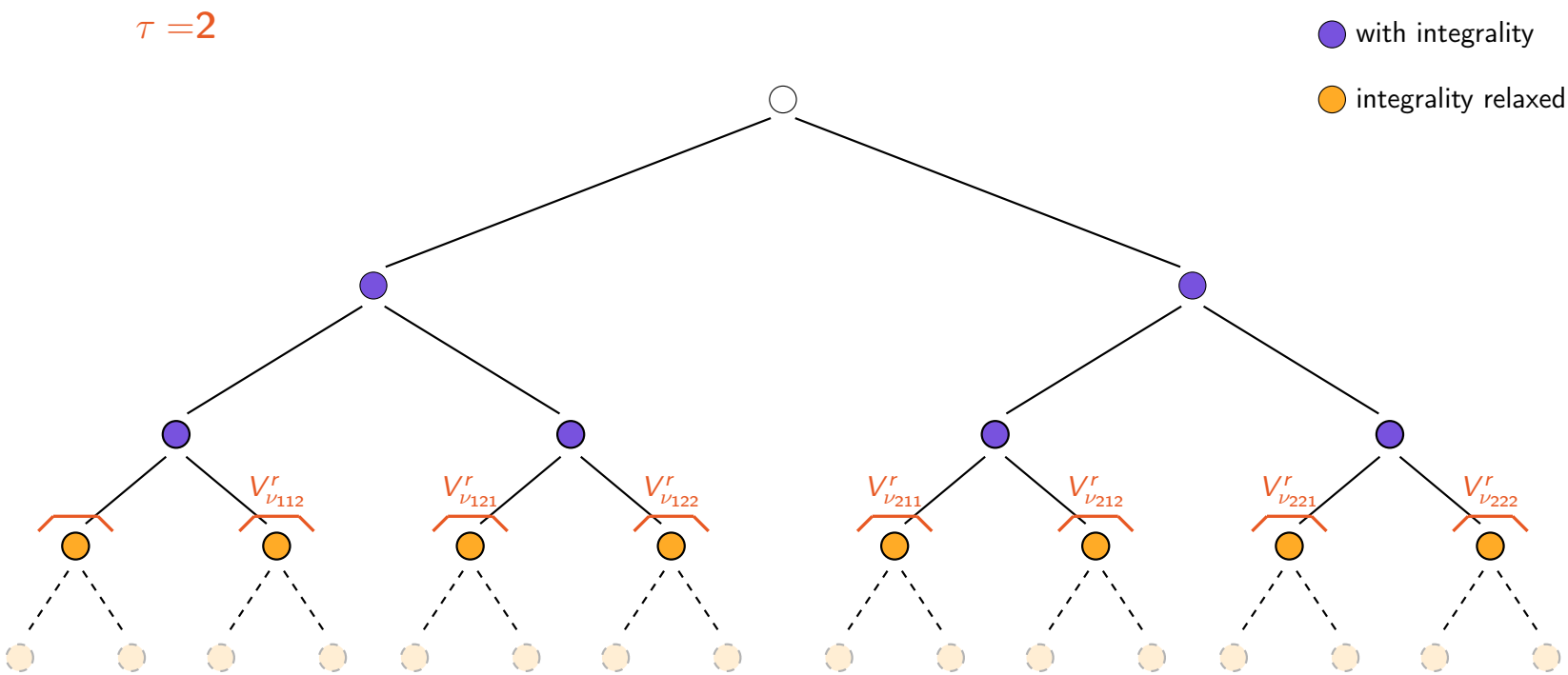

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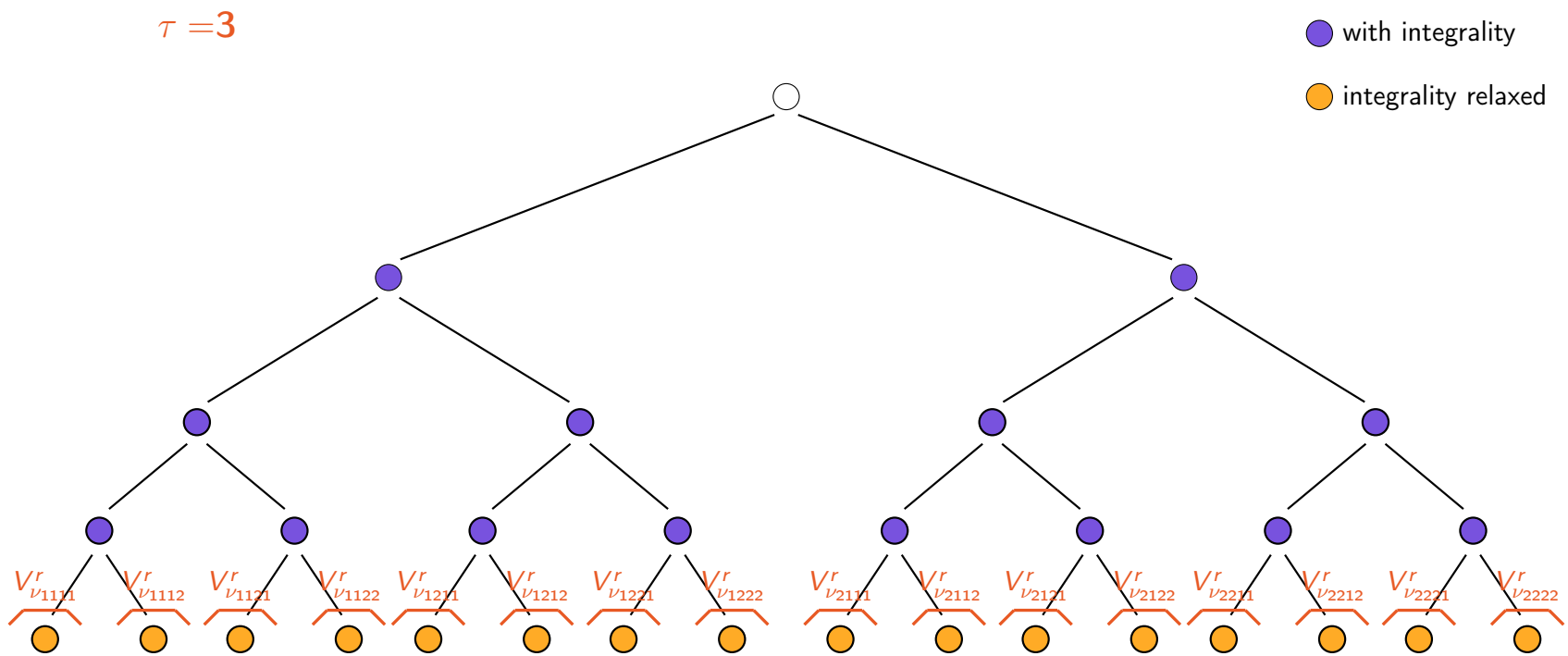

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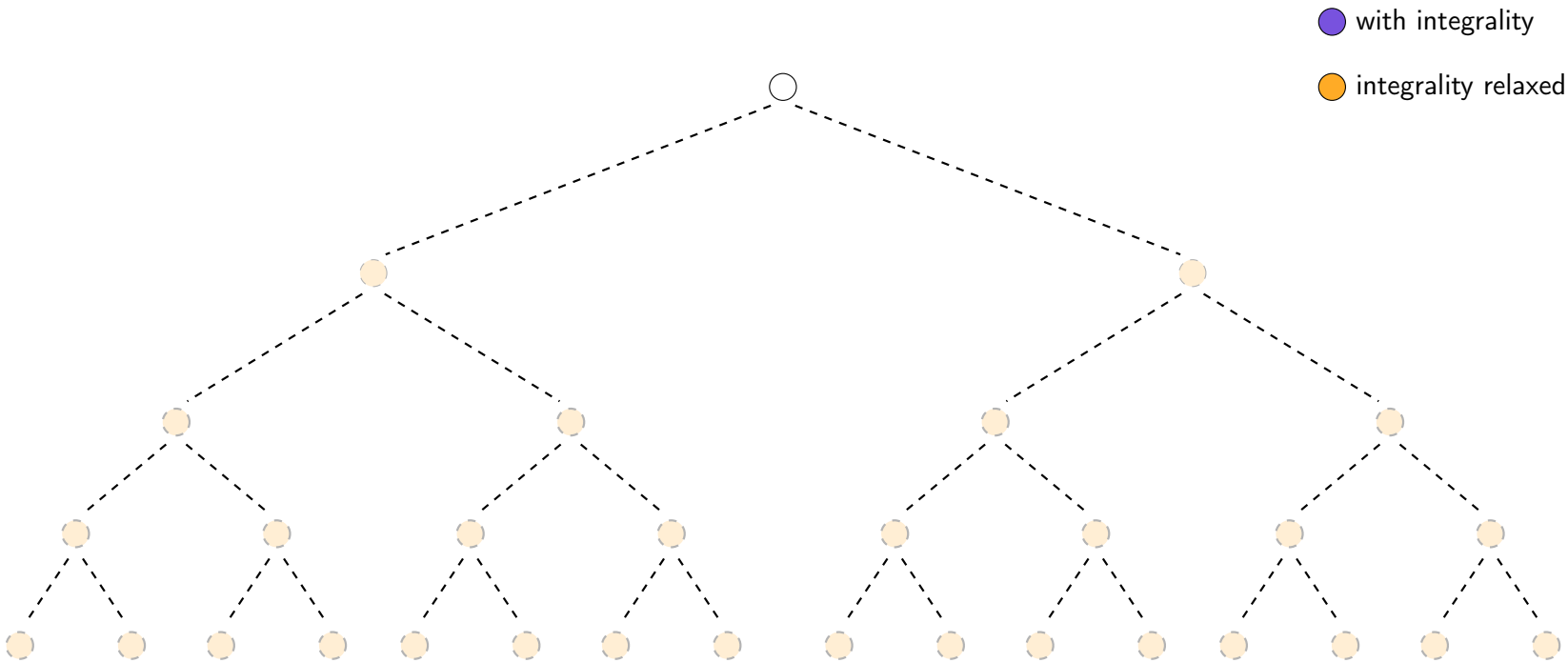

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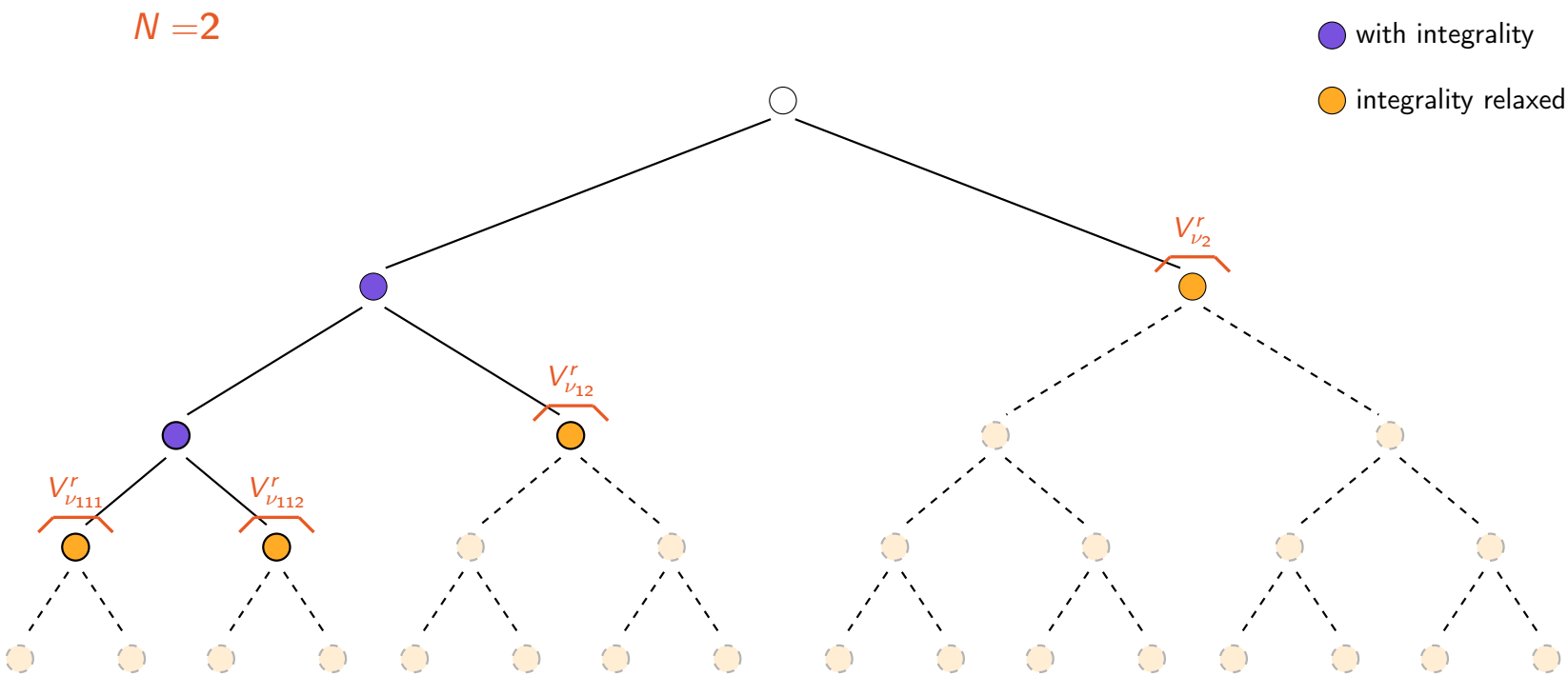

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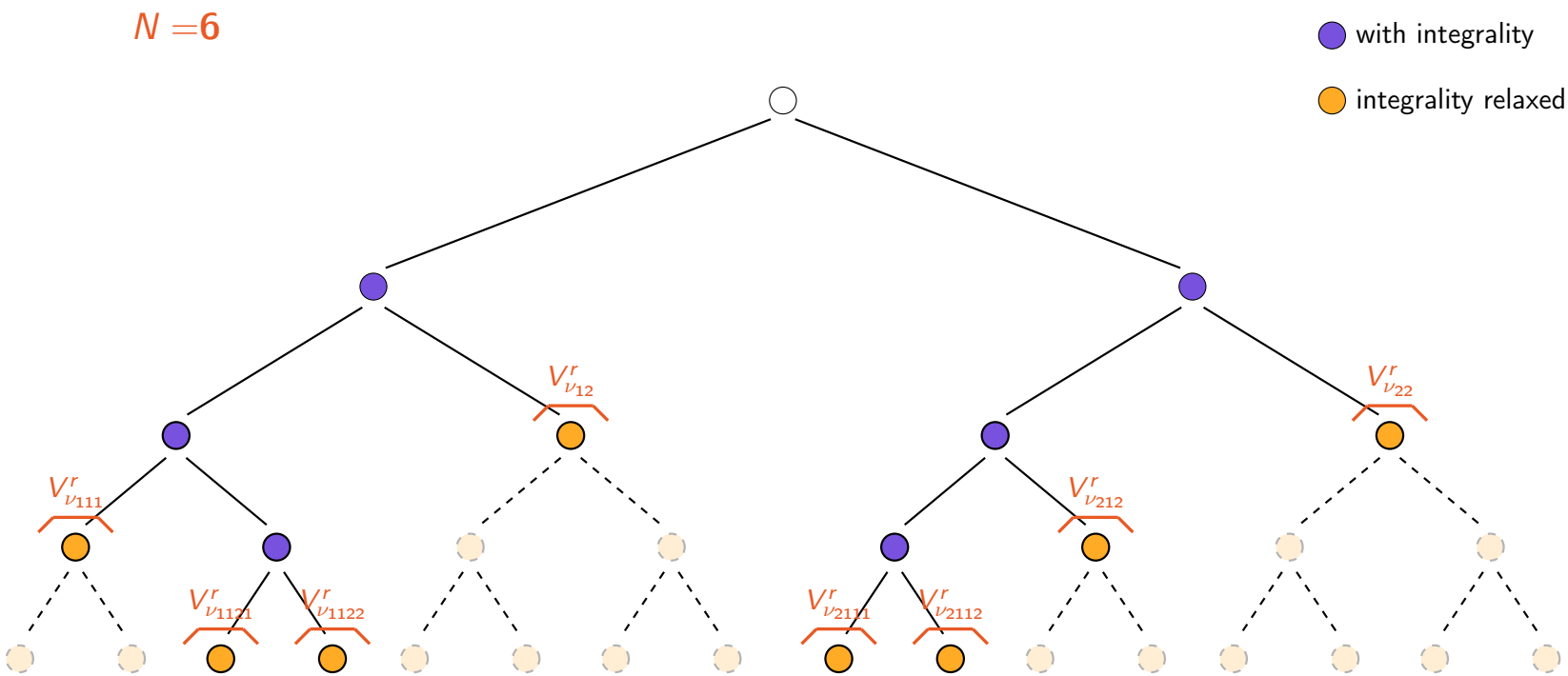

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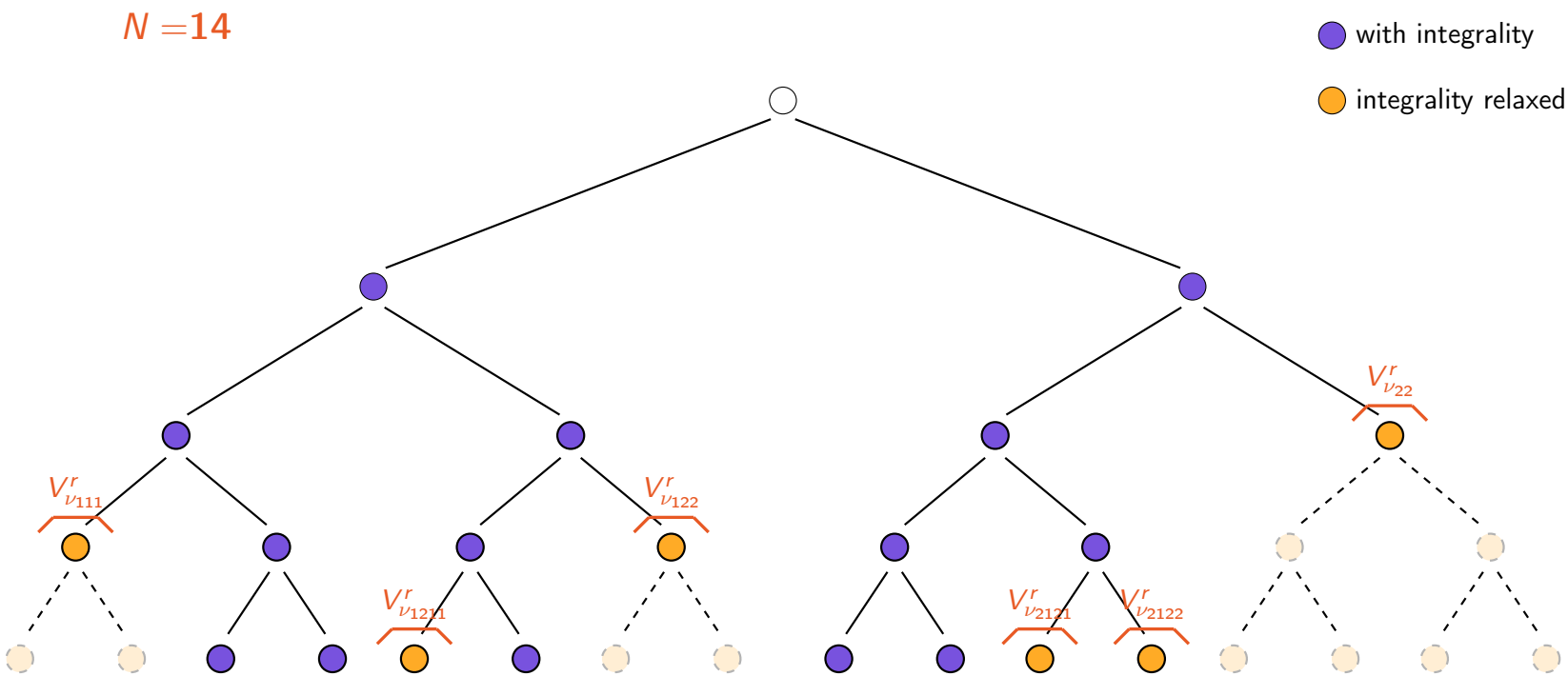

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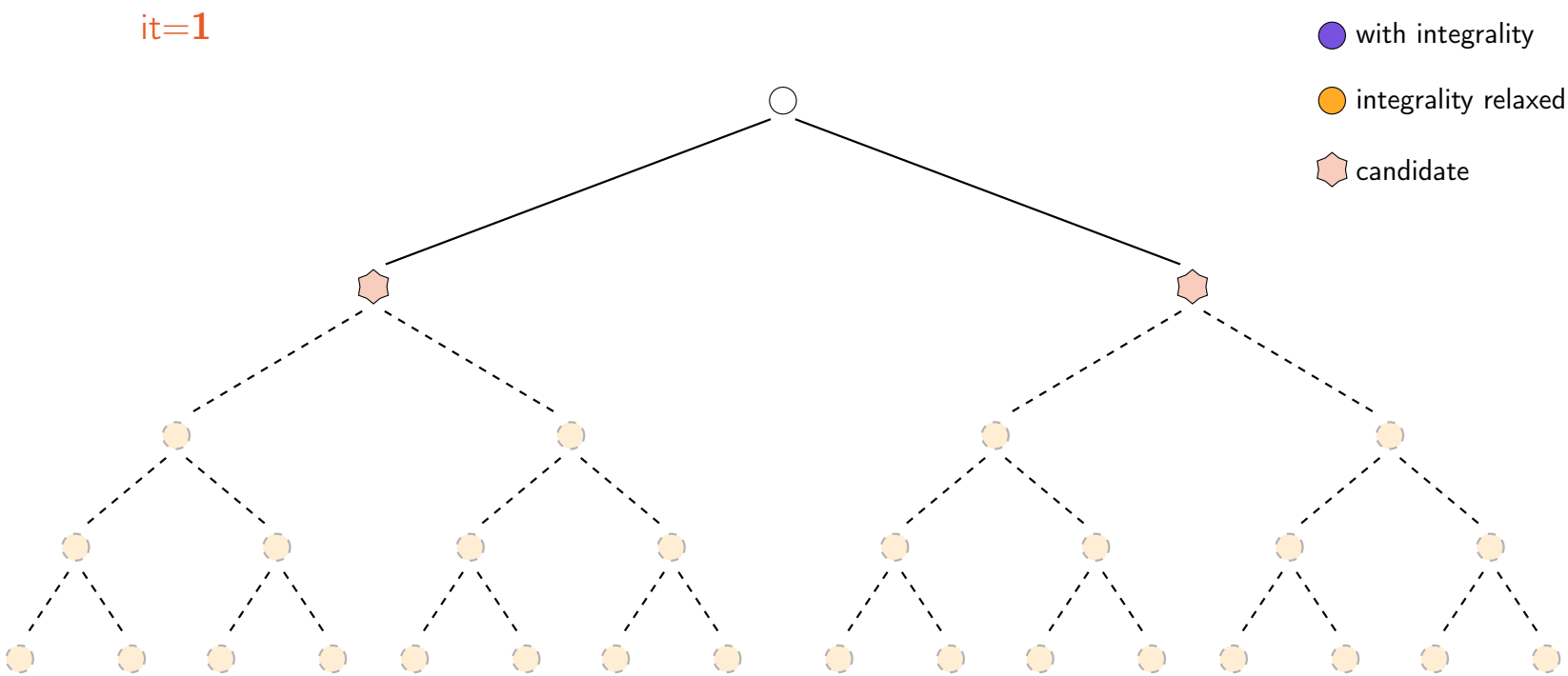

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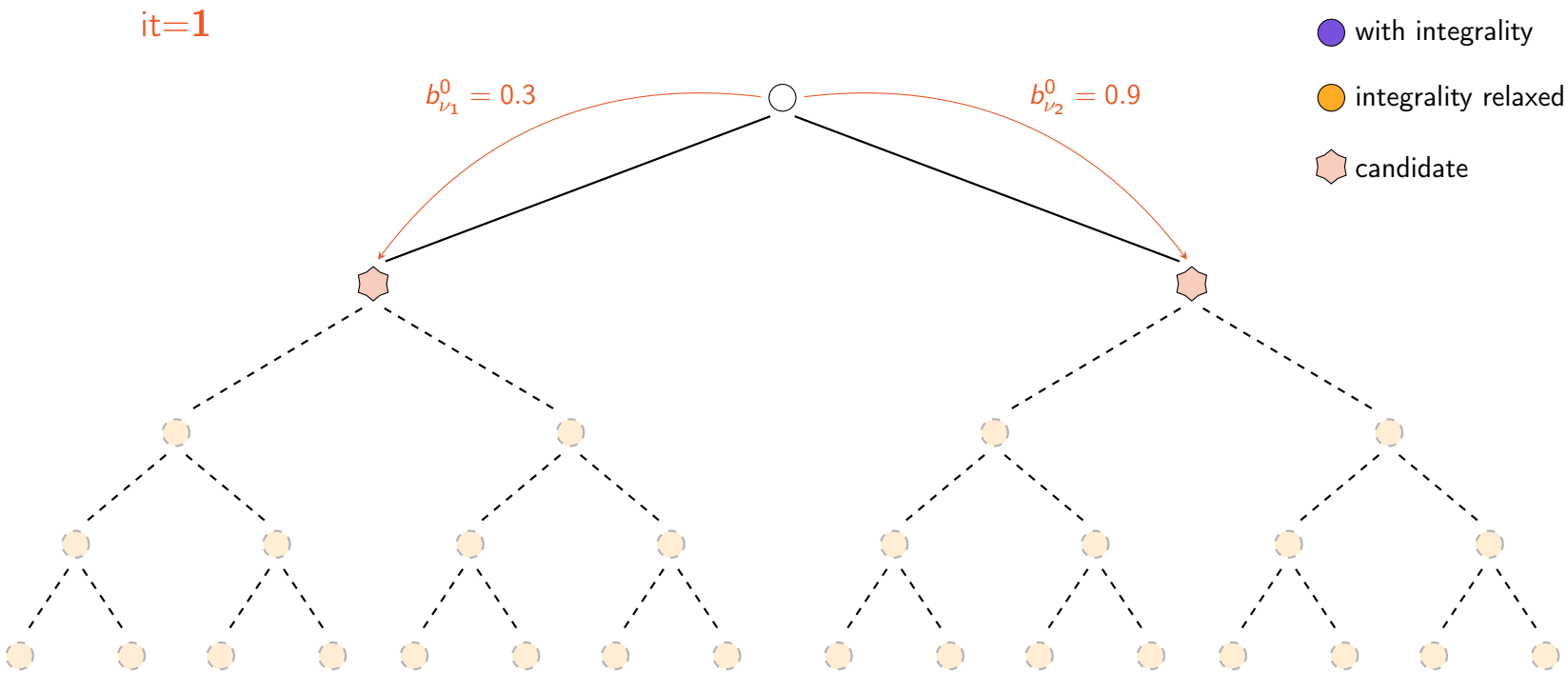

➥ We simulate with SDDP the optimal solution for all candidates.

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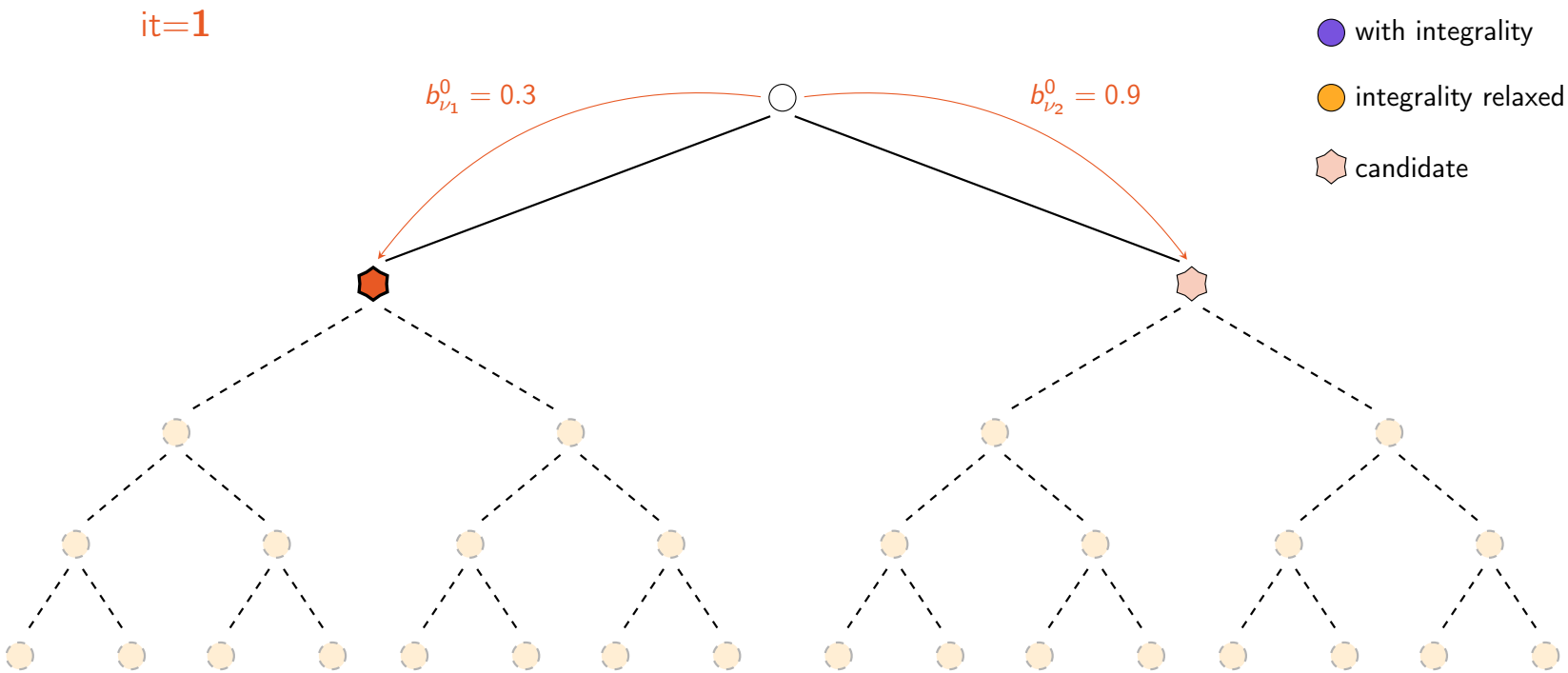

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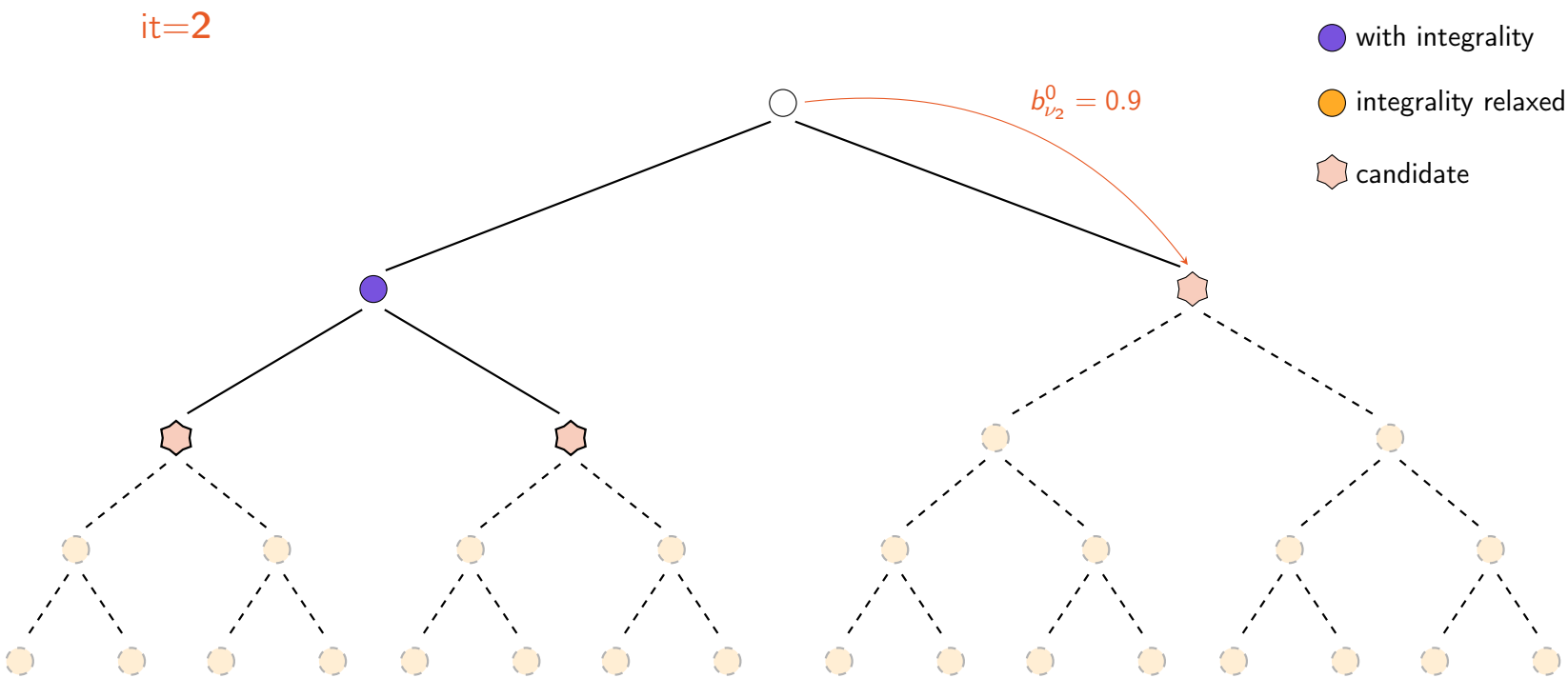

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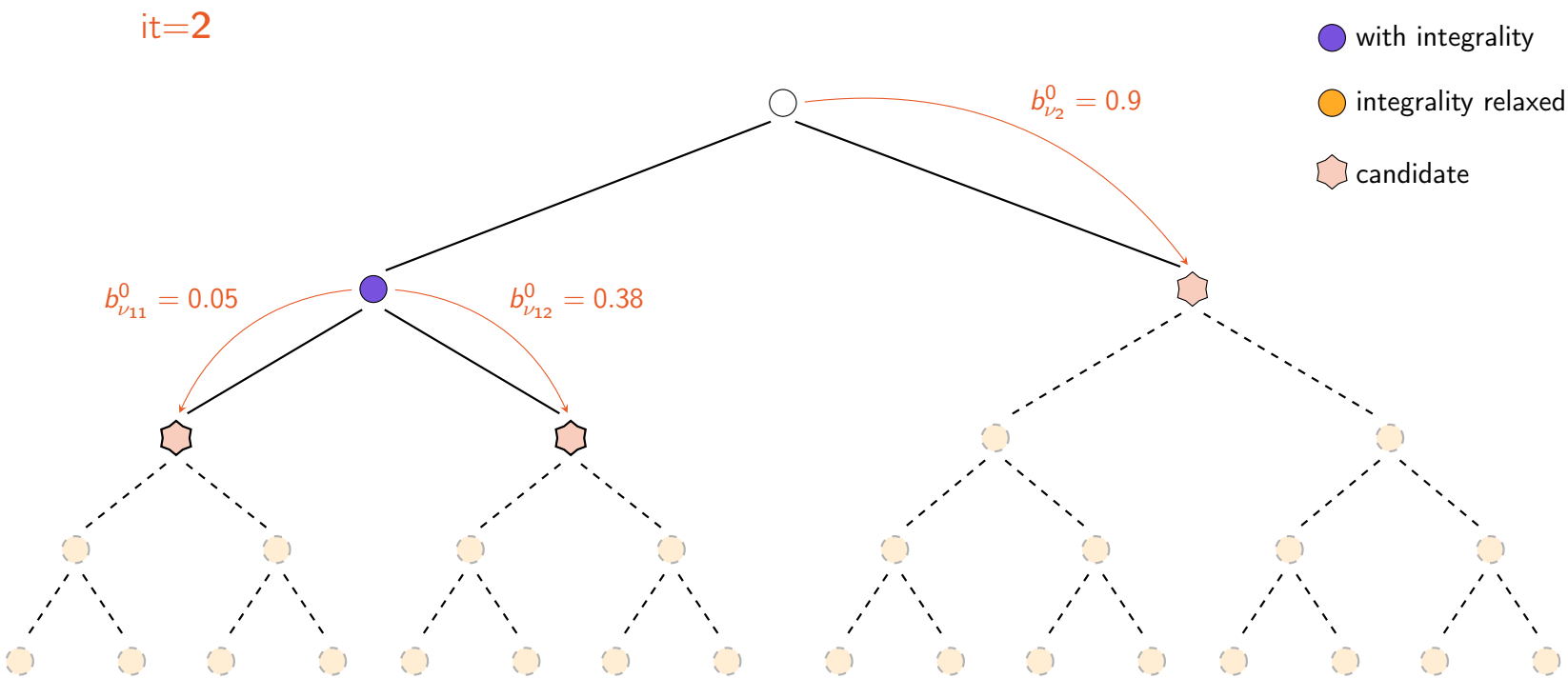

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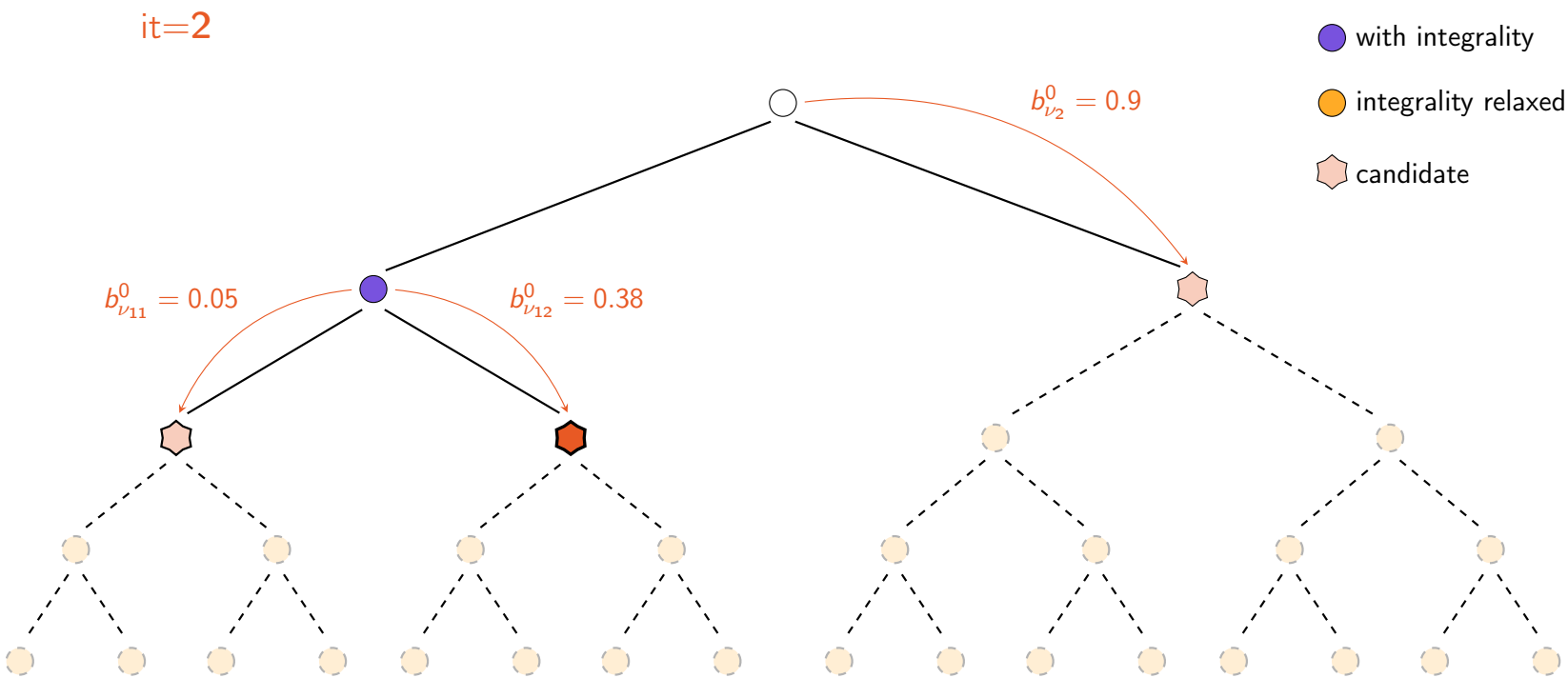

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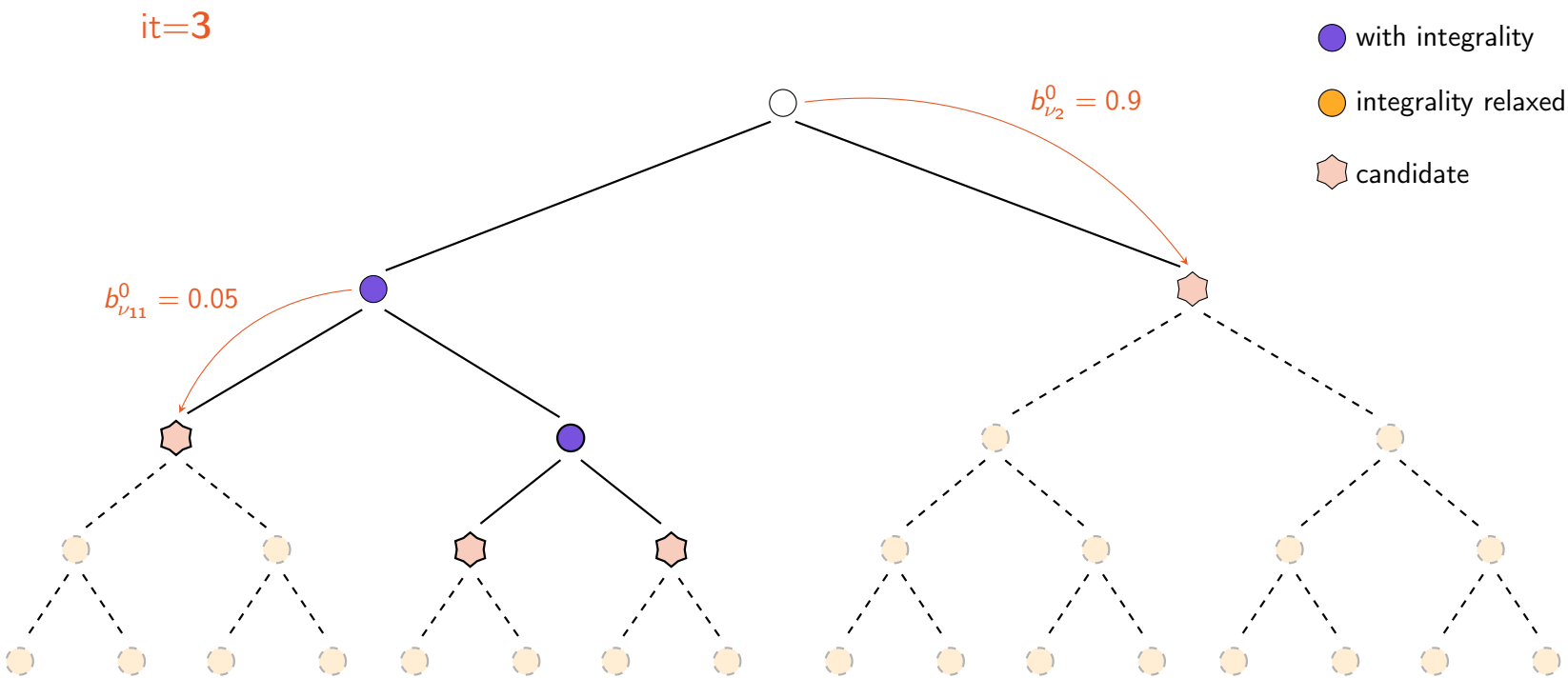

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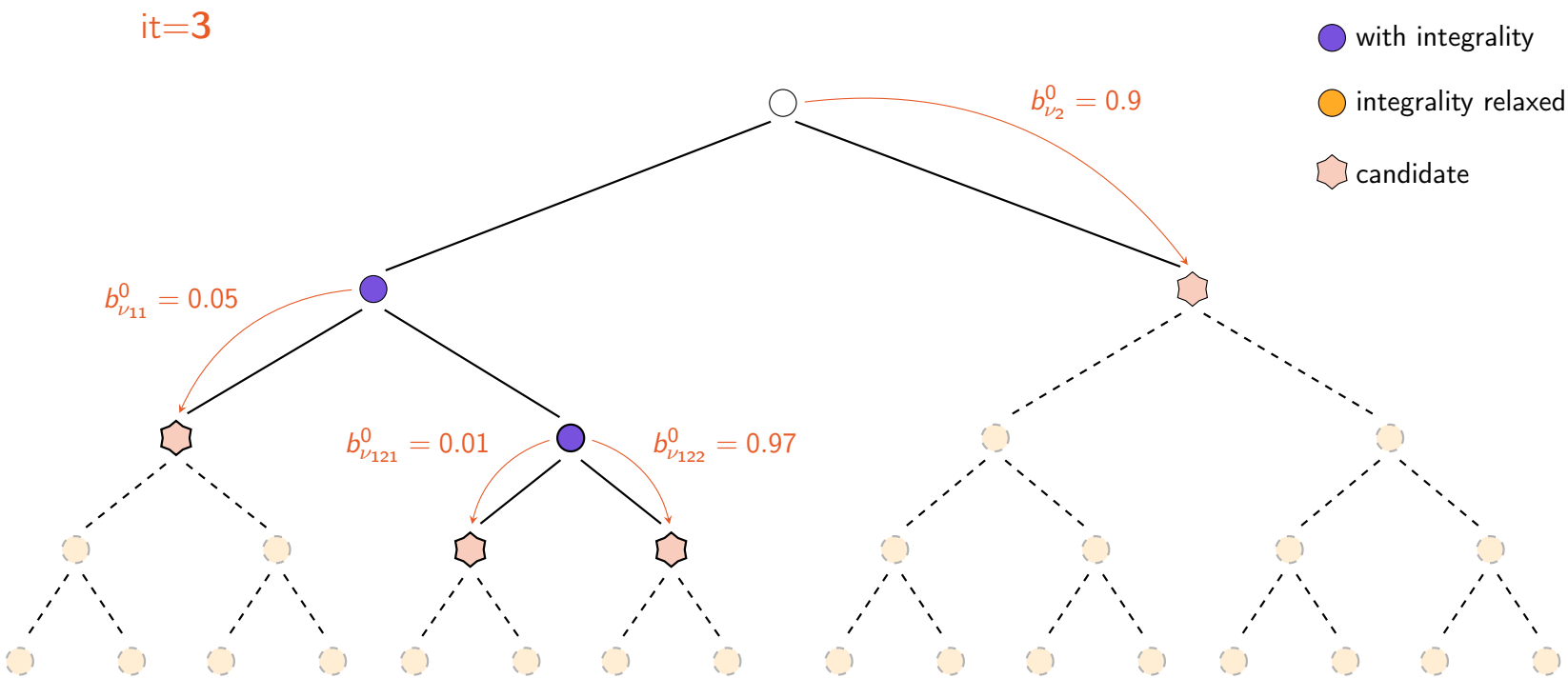

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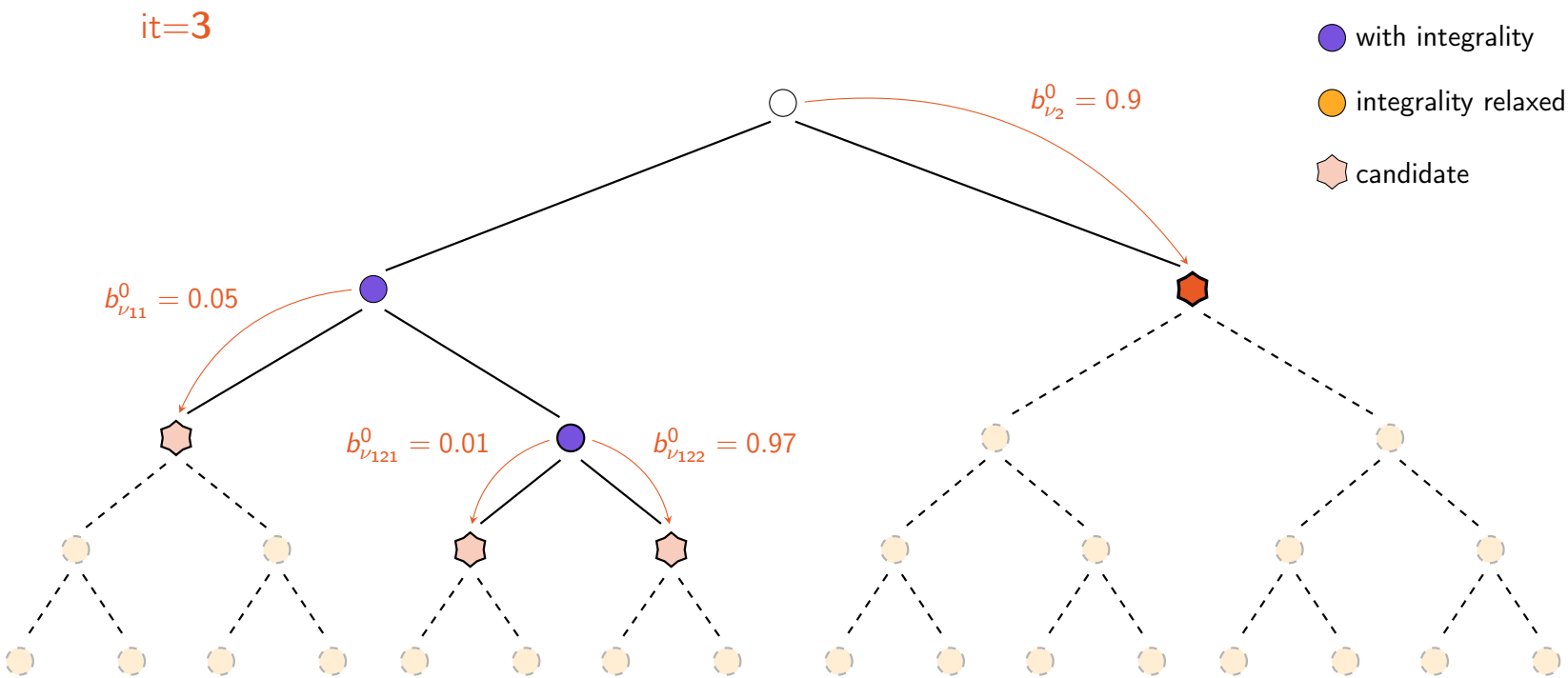

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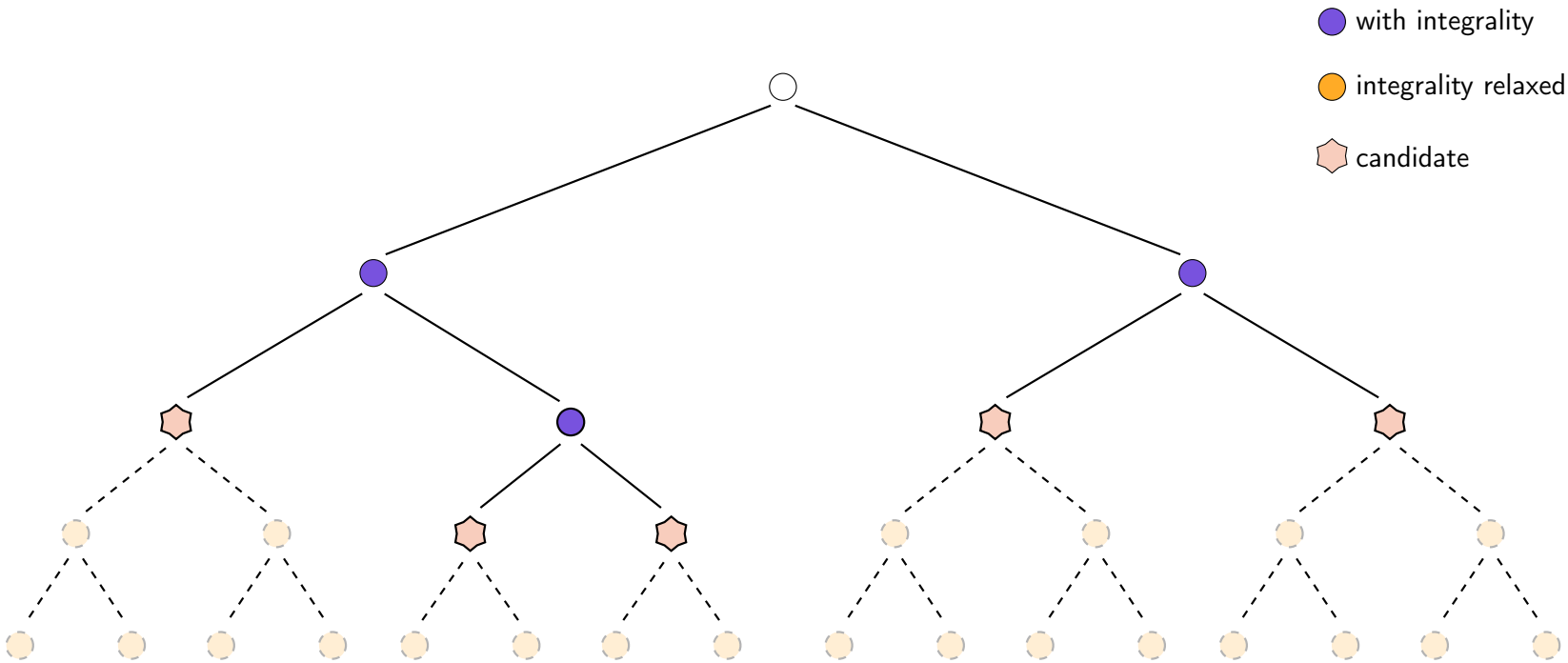

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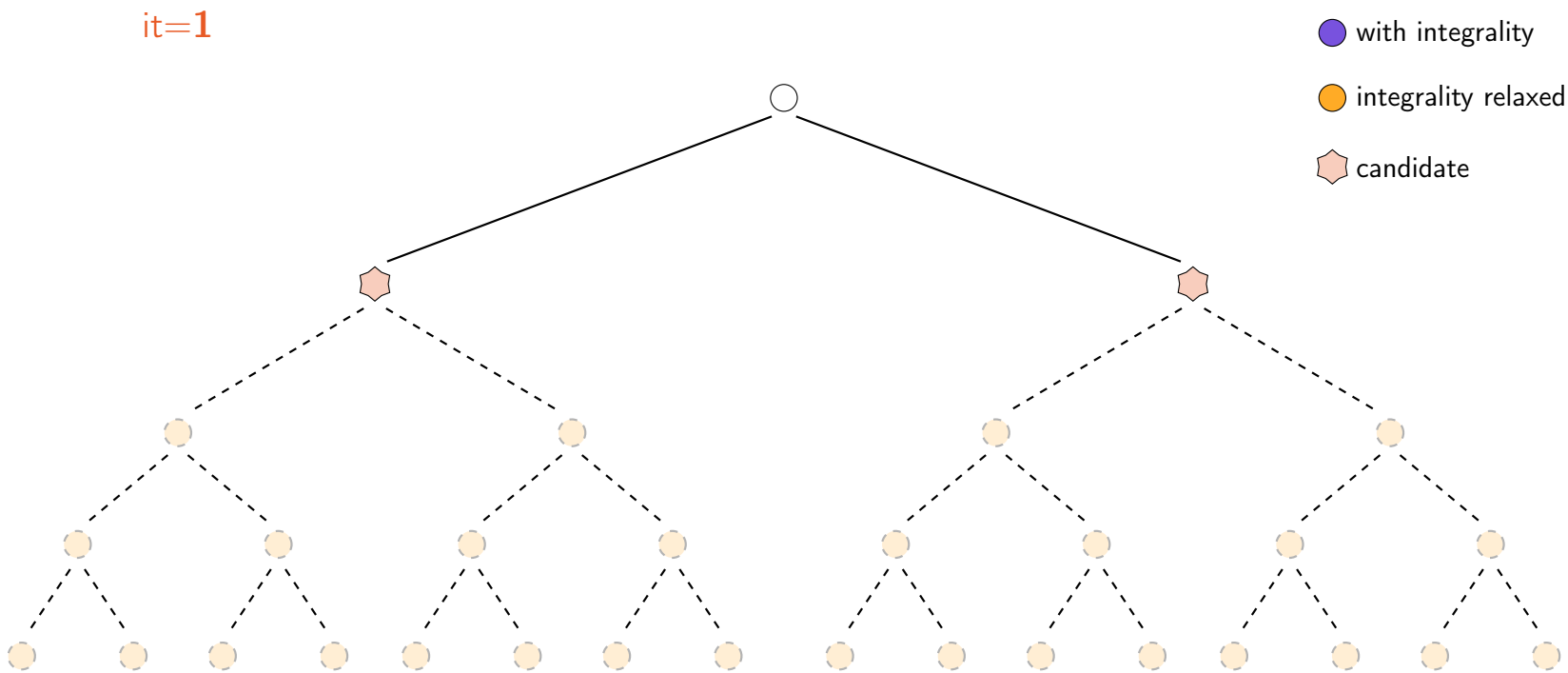

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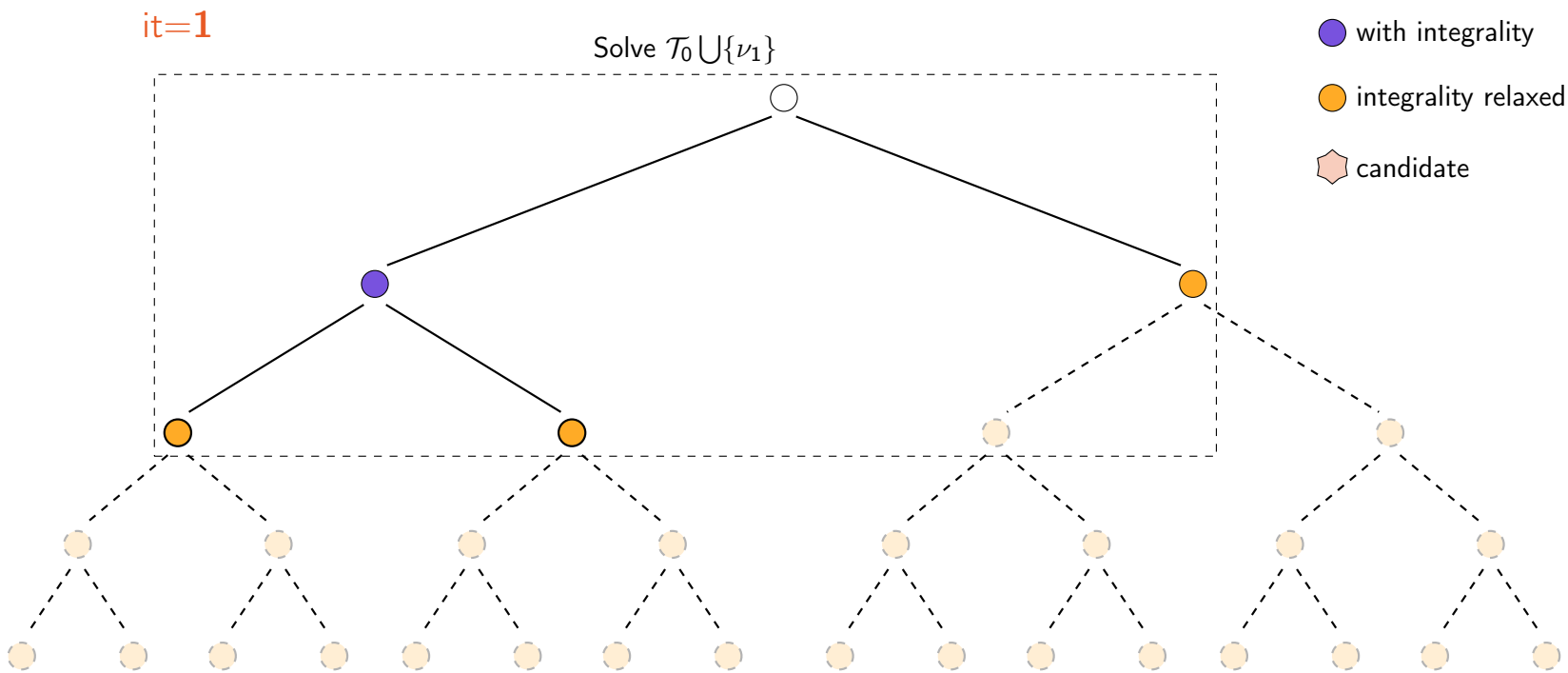

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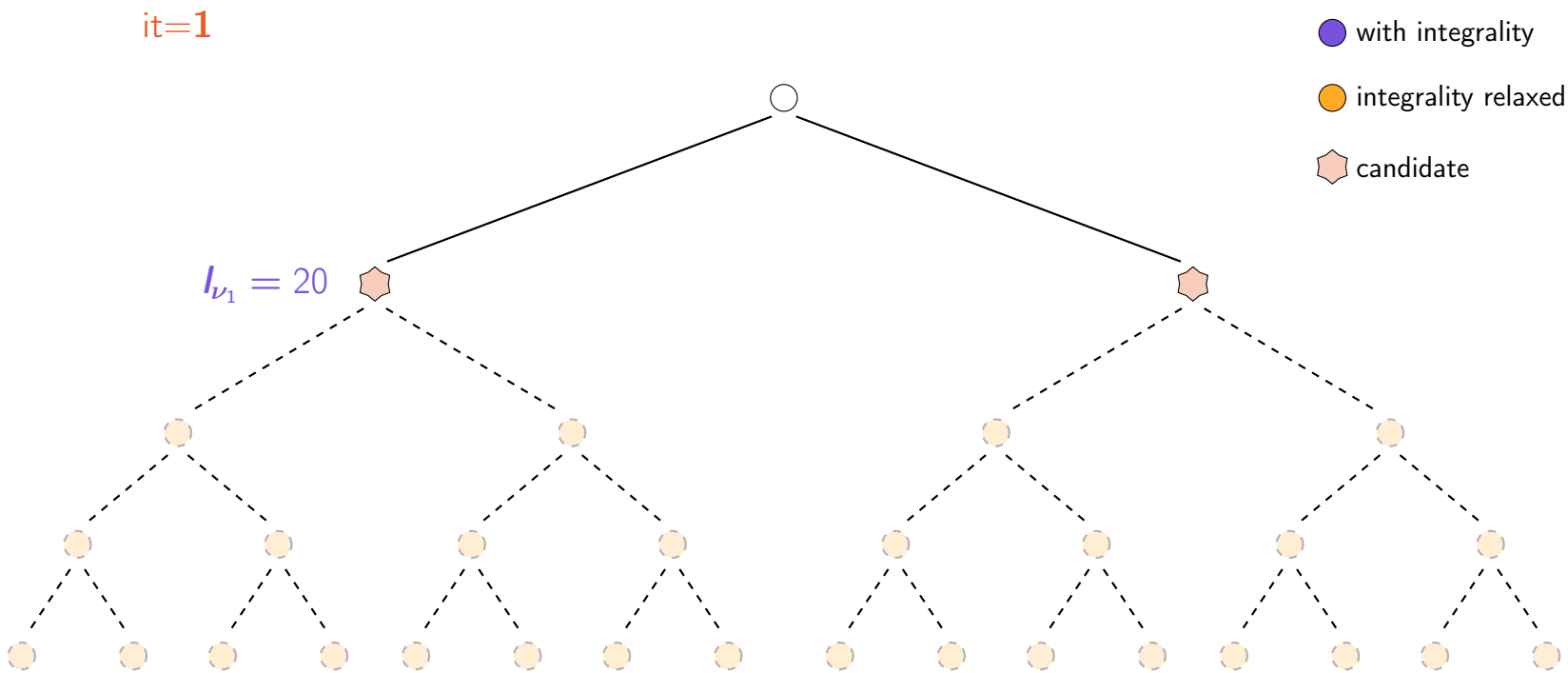

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➥ We obtain lower-bound l_ν

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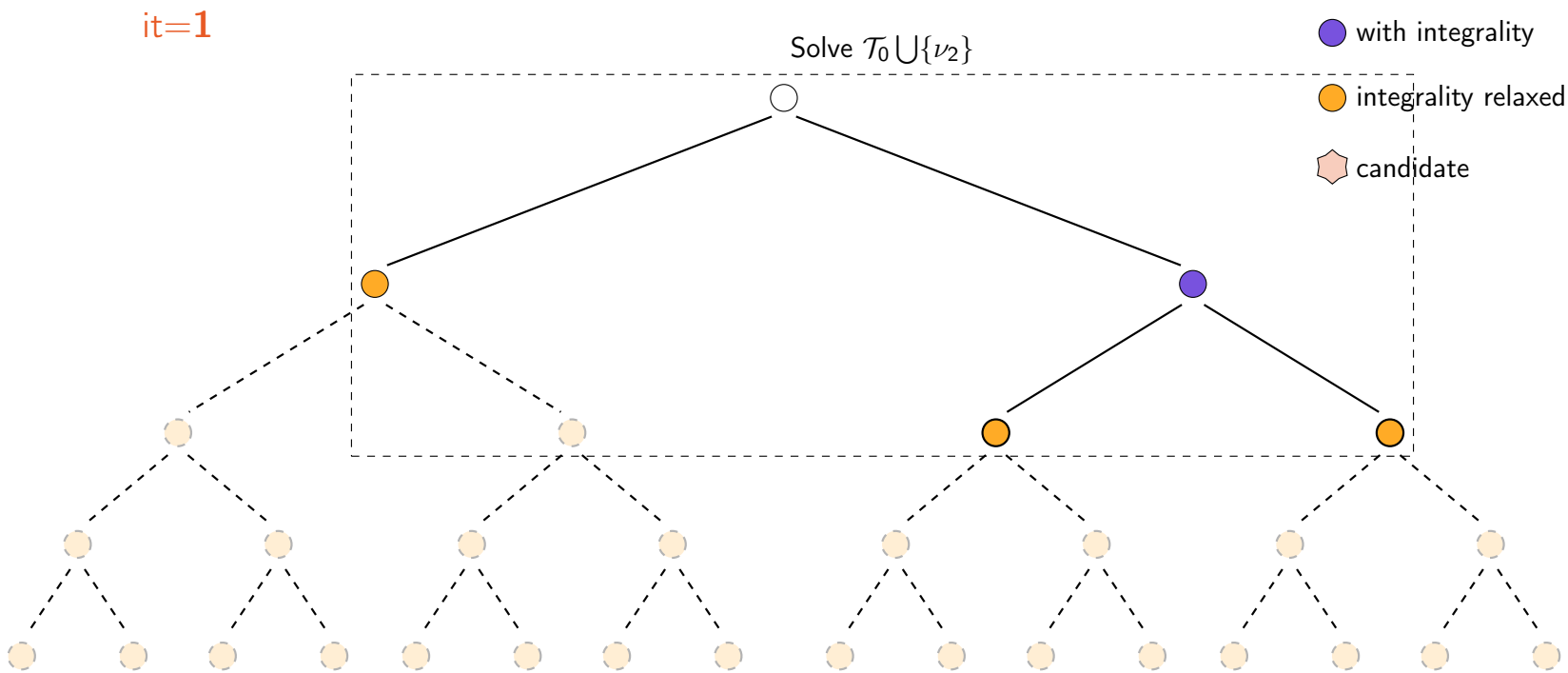

➥ For all candidates ν, we solve the sub-problem on $\mathcal{T}_i \bigcup \{\nu\}$

➥ We obtain lower-bound I_ν

HOW TO GROW THE SUB-TREE?

1. We add time step per time step
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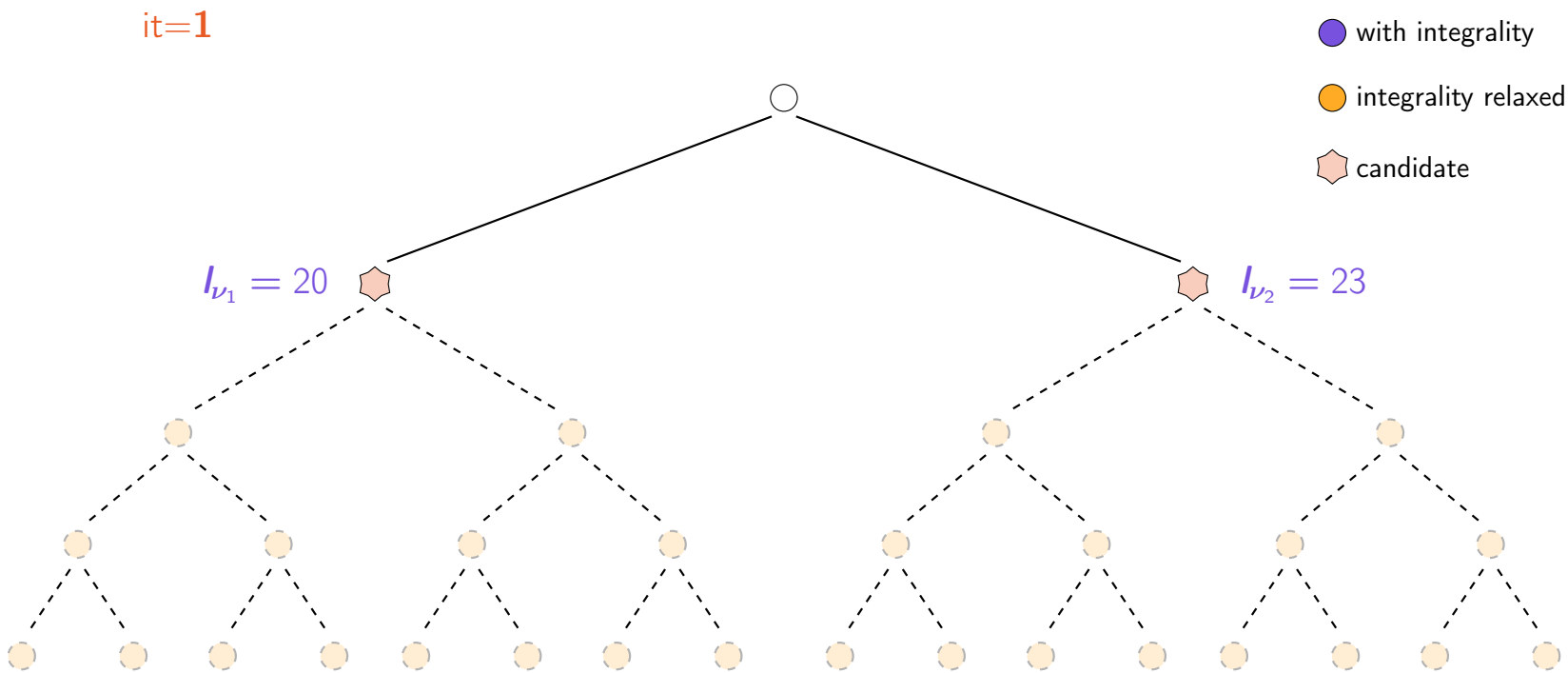

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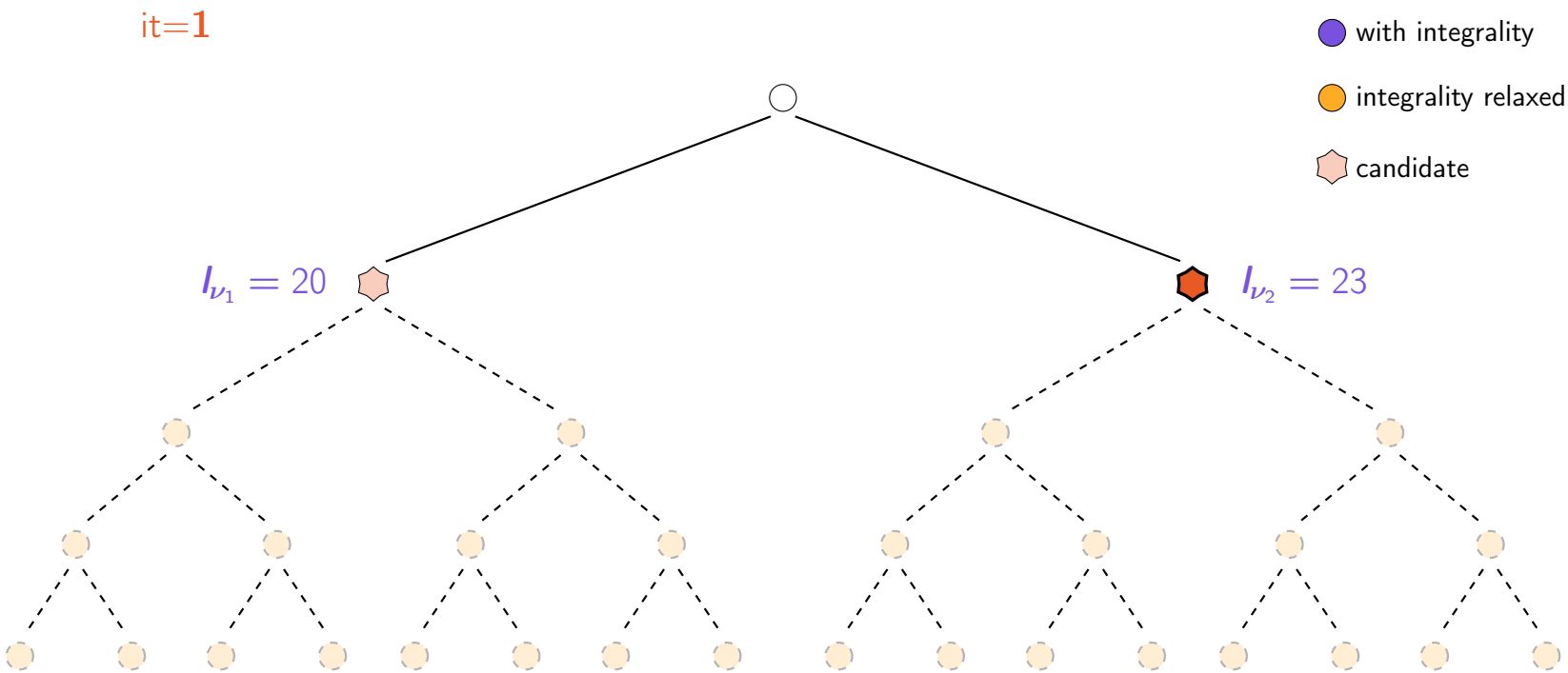

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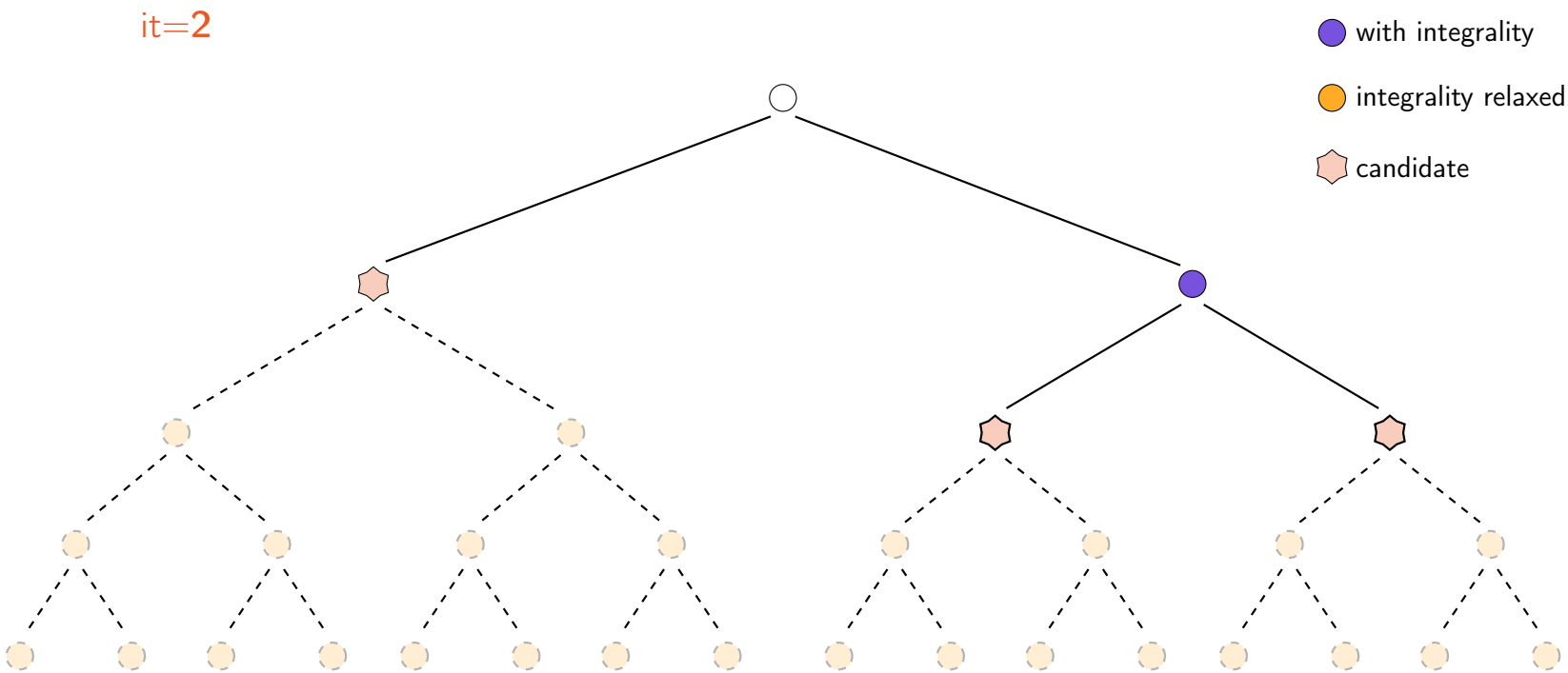

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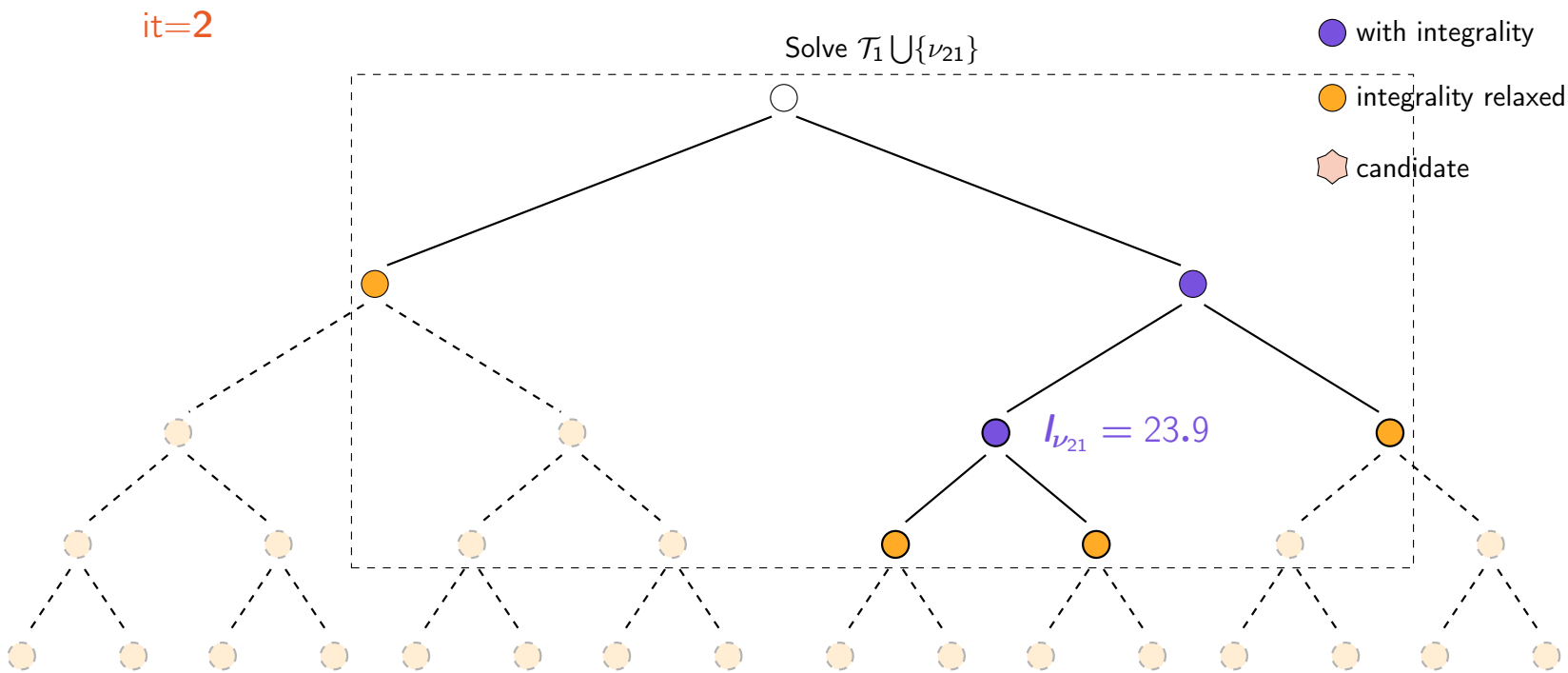

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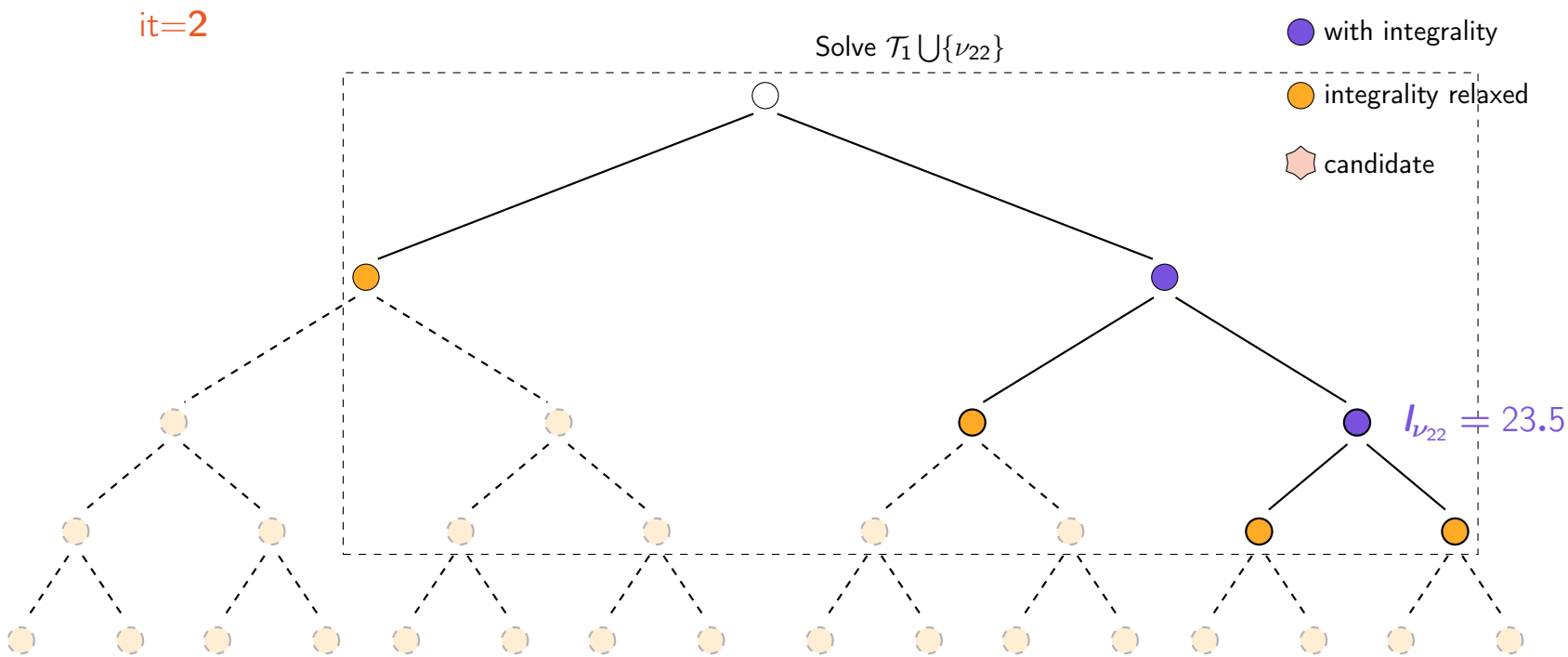

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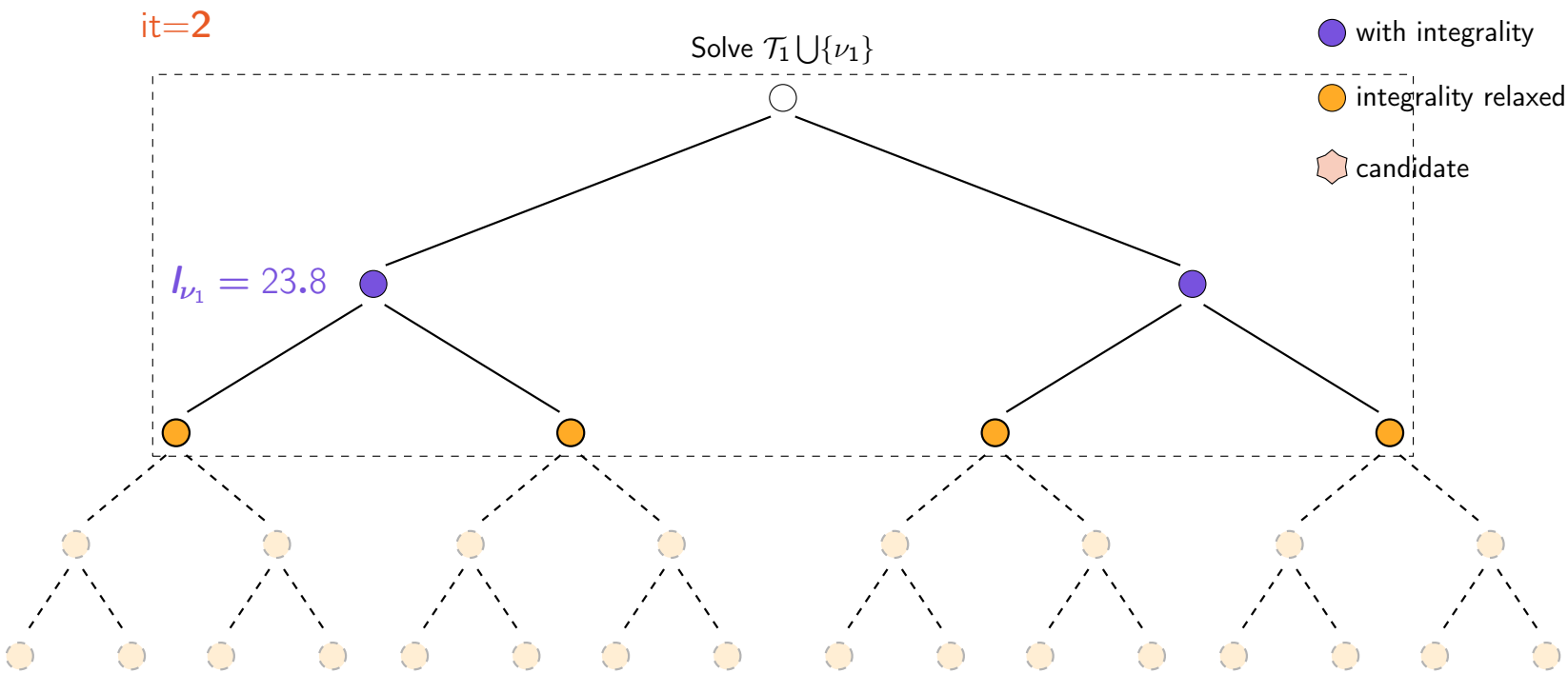

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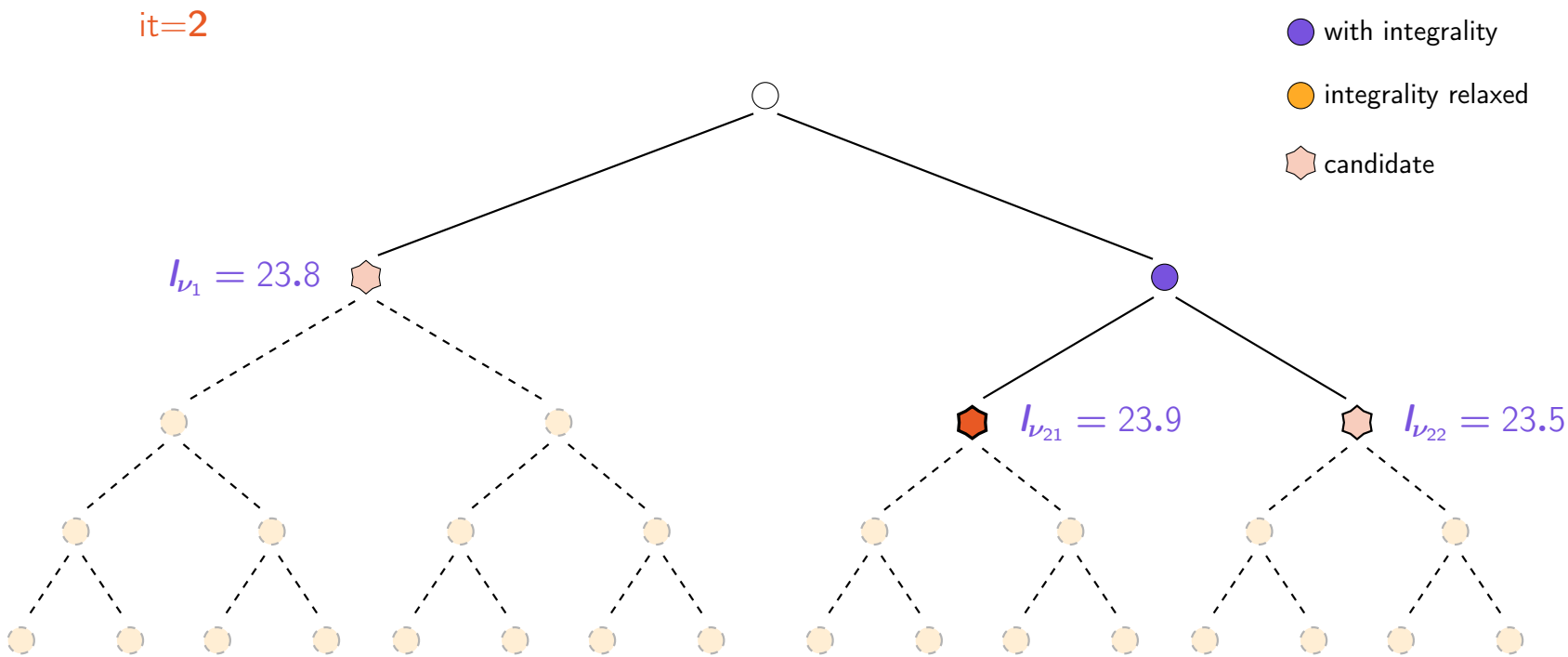

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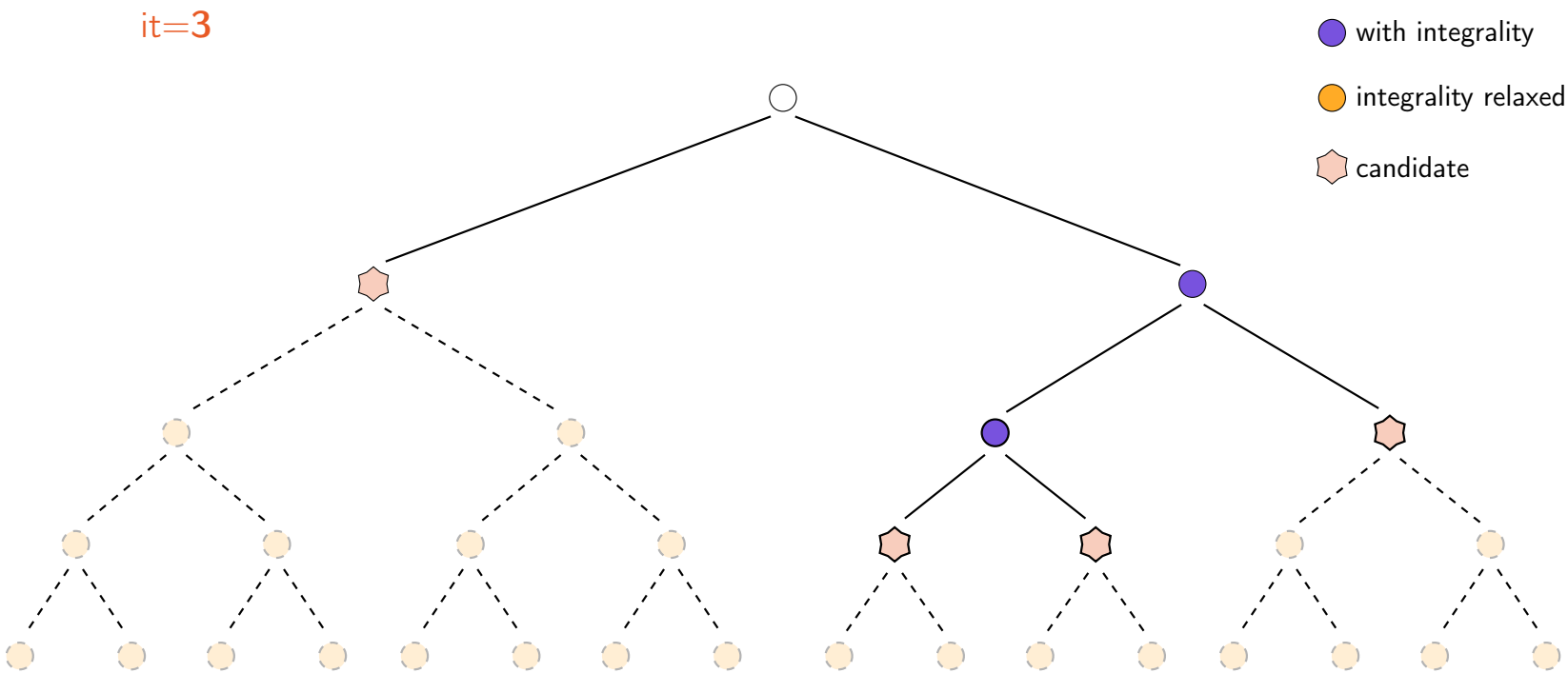

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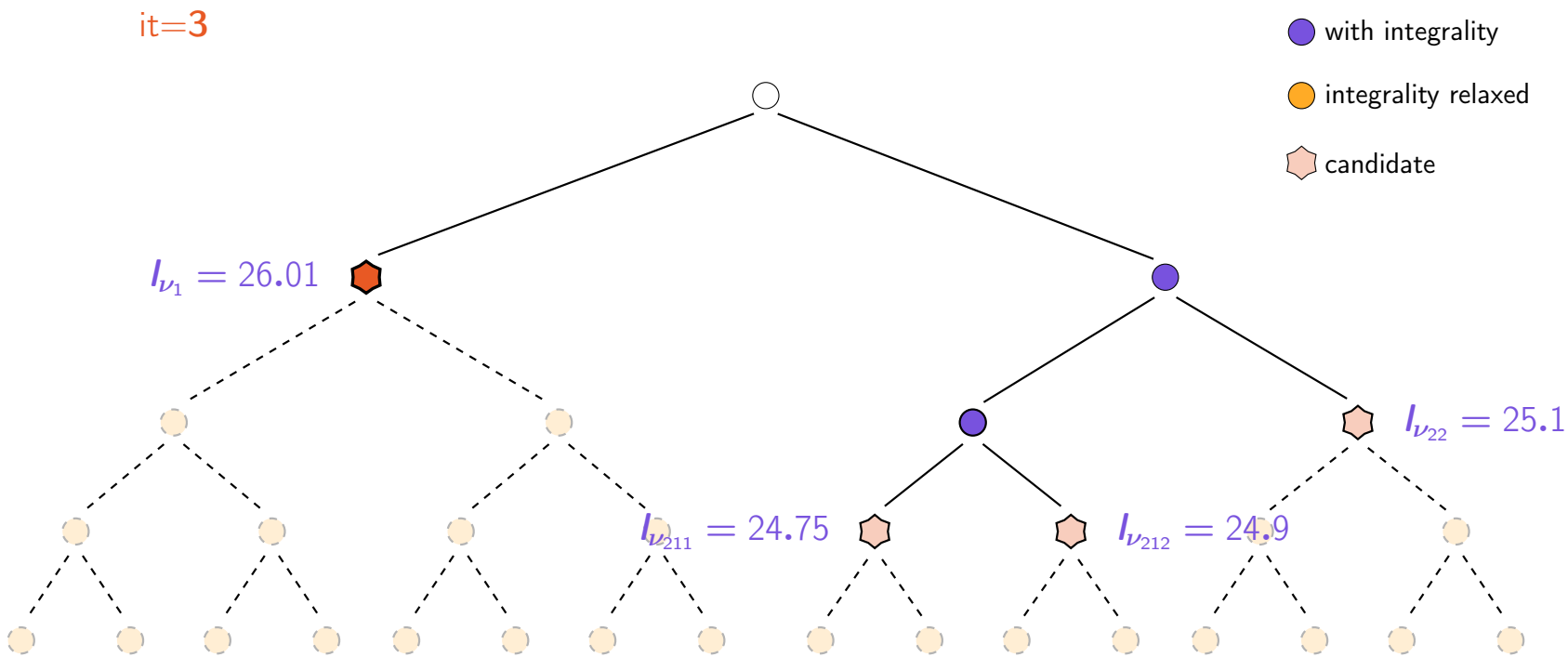

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PRESENTATION OUTLINE

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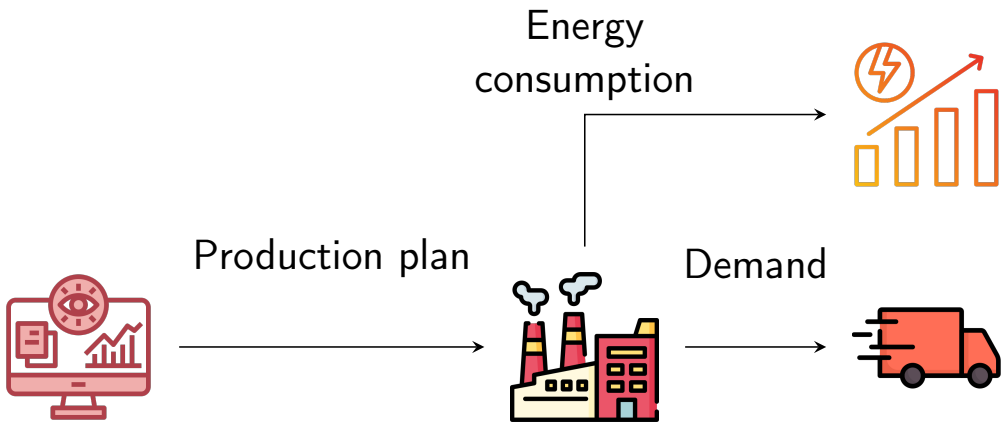
APPLICATION MODEL



$\mathcal{I}$ machines, $\mathcal{J}$ products, a battery

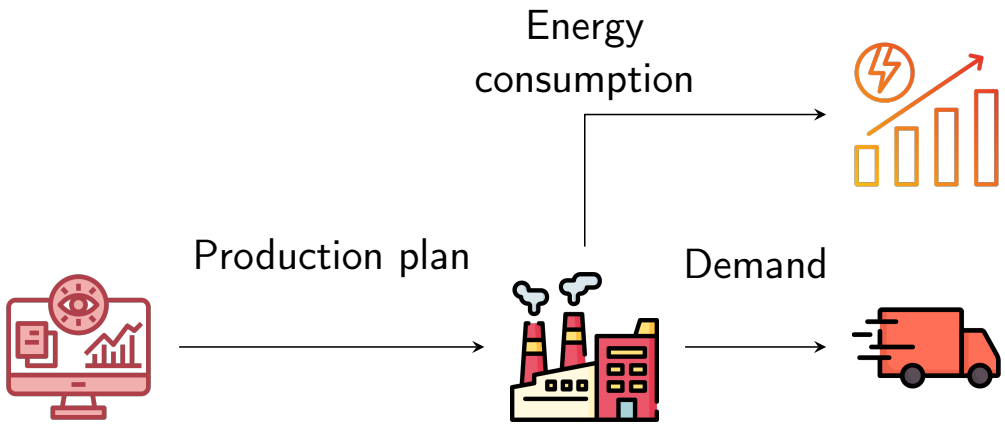
State variable

1. s_t^j: stock of j at t
2. soc_t: energy in the battery at t

Controls

1. $b_t^j = \begin{cases} 1 \text{ if we produce } j \text{ at } t \text{ on } i, \\ 0 \text{ otherwise.} \end{cases}$
2. u_t^{ij}: production of j on i at t
3. q_t^{grid}: energy bought at t
4. ϕ_t^+ and ϕ_t^-: energy charged/discharged at t

APPLICATION MODEL



$\mathcal{I}$ machines, $\mathcal{J}$ products, a battery

Process and Physical constraints

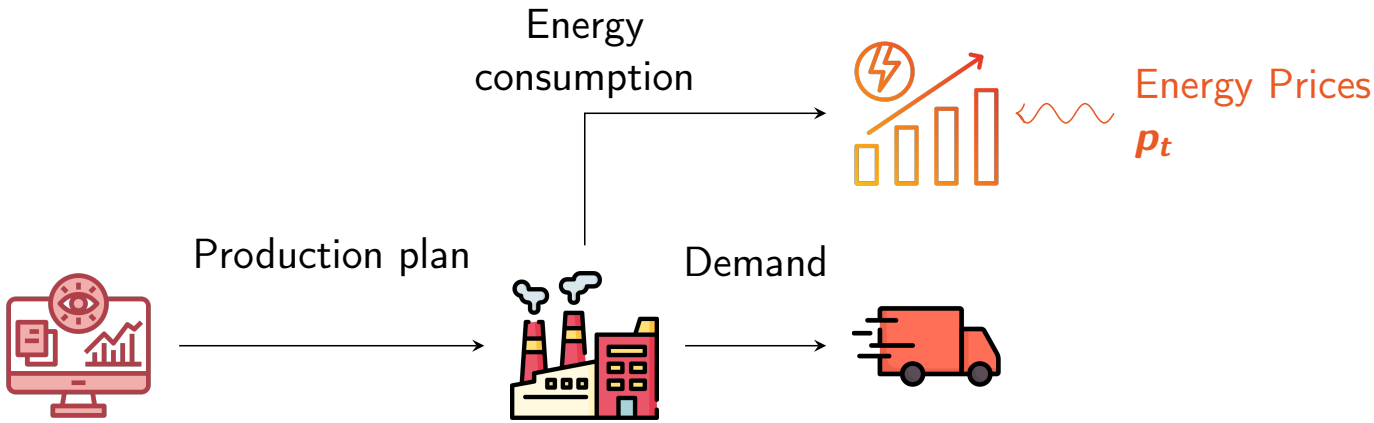
Dynamics

1. $s_t^j = s_{t-1}^j - d_t^j + \sum_i u_t^{ij}$
2. $soc_t = soc_{t-1} - \frac{1}{\eta}\phi_t^- + \eta\phi_t^+$

Production Constraints

1. $\underline{u^i}\, b_t^{ij} \leq u_t^{ij} \leq \overline{u^i}\, b_t^{ij}$
2. $\sum_{j\in\mathcal{J}} b_t^{ij} \leq 1$
3. $\phi_t^- - \phi_t^+ + q_t^{\text{grid}} \geq \sum_{i,j} f^{ij}(u_t^{ij})$
4. $\max_i b_t^{ij} + \max_i b_t^{ik} \leq 1$ if $j,k \in \mathcal{E}$

APPLICATION MODEL



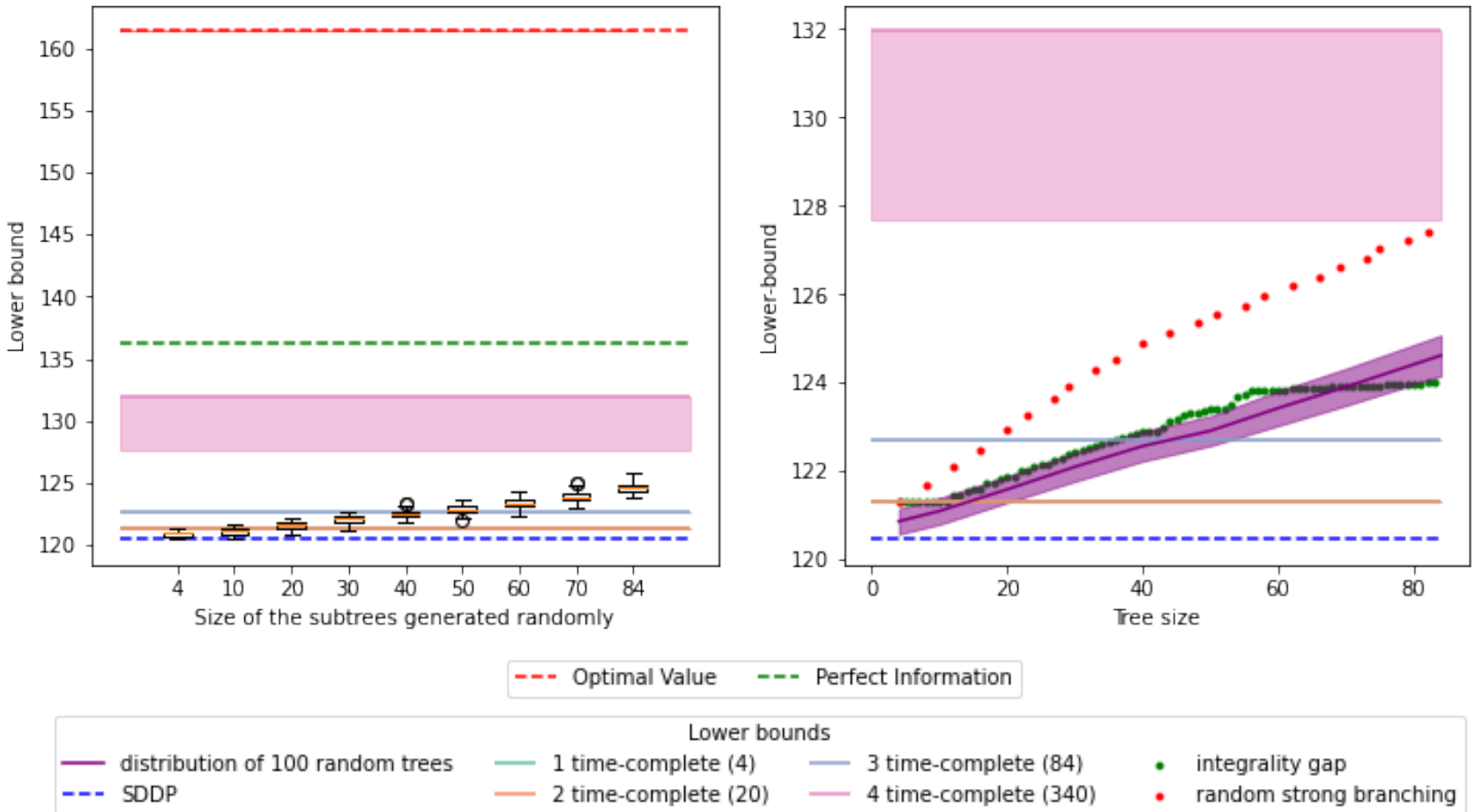
$\mathcal{I}$ machines, $\mathcal{J}$ products, a battery

Objective

$$\mathbb{E}\left[\sum_{t=1}^{T} p_t \, q_t^{\text{grid}}\right]$$

PRESENTATION OUTLINE

$T = 5, Q = 4, I = 3, J = 3$



- The **strong-branching generation improves the most the lower-bound**.
- Integrality-gap does better than random generation until a certain point, and then stagnates.
- **All lower-bounds are far from the optimal value** (at best -19%).

$T = 5, Q = 4, I = 3, J = 3$

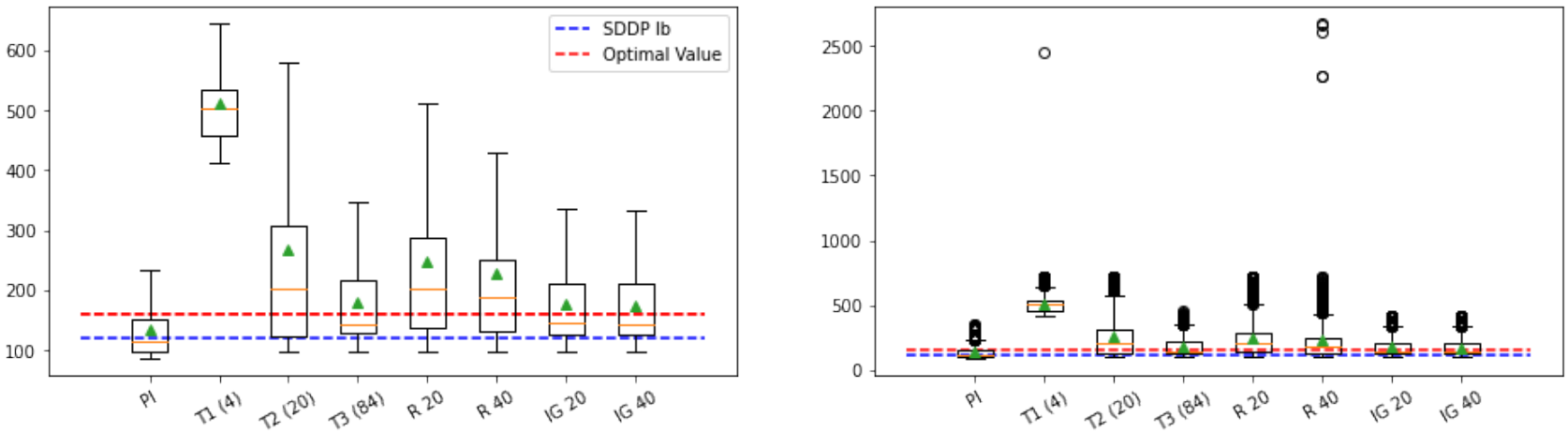


Figure: Simulations with different methods over all scenarios

- Unfeasiblity is **highly penalized**
- Simulation results are **encouraging**.
- The **gap subtrees yield the best results**.
- Bear in mind, these are **no general conclusions**.

PRESENTATION OUTLINE

State-of-the-art

Lower approximations of MiSLP

Numerical Results

Improving performances

IMPROVING PERFORMANCES

We have constructed a sub-tree $\mathcal{T}_i$.

How can we improve the solution of the partially relaxed problem $(P_{x_0}^{\mathcal{T}_i})$?

1. Improve the solution of the MILP
   - ➥ add specific lot-sizing cuts
2. Improve the cost-to-go approximation used at the leaves of the sub-tree
   - ➥ use some of SDDiP cuts (strengthened Benders' cuts, lagrangian cuts)
3. Add some nodes with continuously relaxed variables but valid cuts

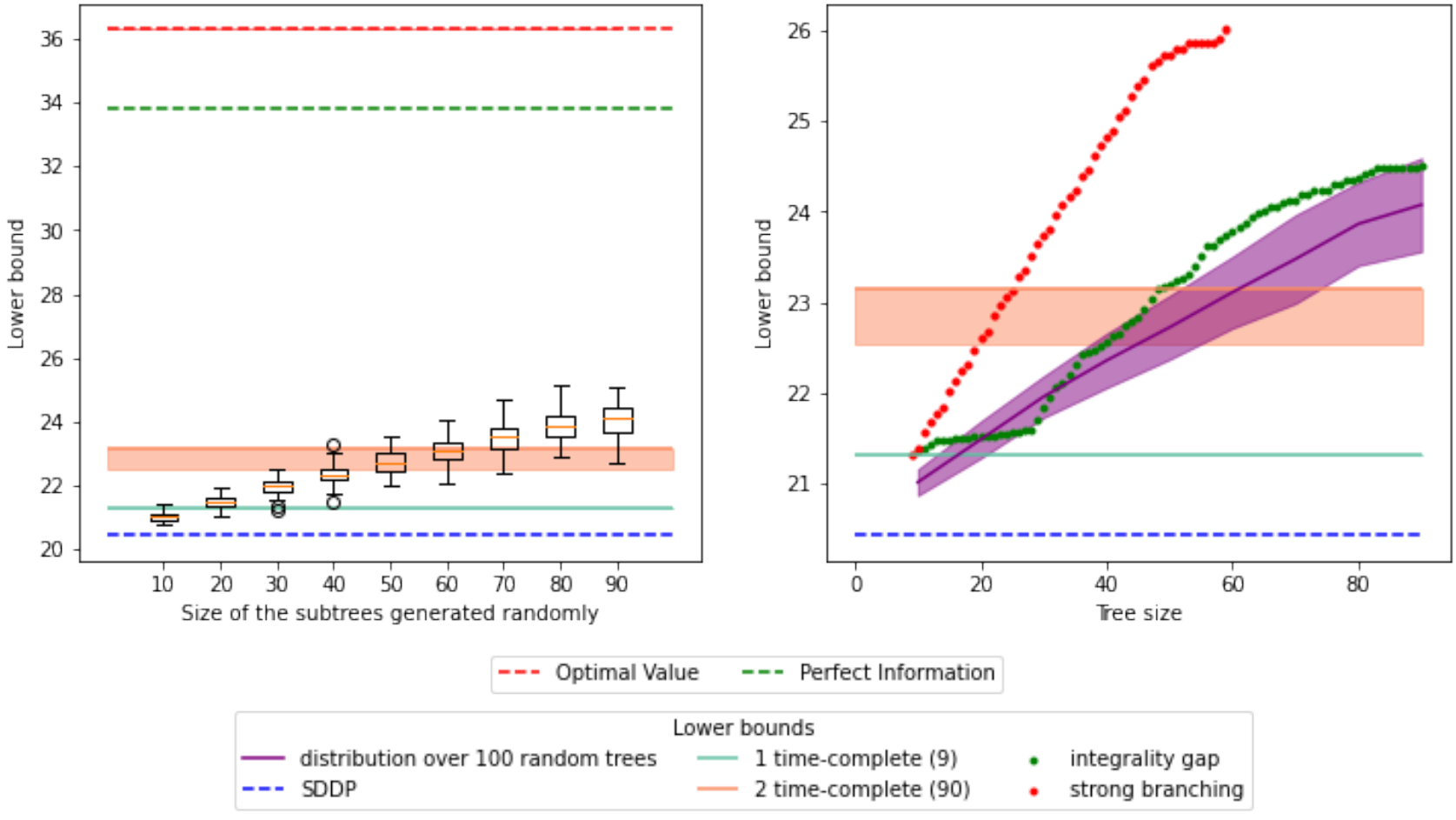
IN A NUTSHELL

- Depending on the method used to generate a sub-tree, we **can improve the lower-bounds obtained and the policies**.
- Preliminary results are **encouraging** to get good policies for MiSLP.
- Maybe **exploiting the structure of the problem** could lead us to specific generation methods that would perform better.

Thank you for your attention, any questions?

Zoé Fornier, zoe.fornier@enpc.fr

$T = 3, Q = 9, I = 3, J = 4$


Lower bound
Size of the subtrees generated randomly
Tree size
Optimal Value
Perfect Information
Lower bounds
distribution over 100 random trees
SDDP
1 time-complete (9)
2 time-complete (90)
integrality gap
strong branching

$T = 3, Q = 9, I = 3, J = 4$

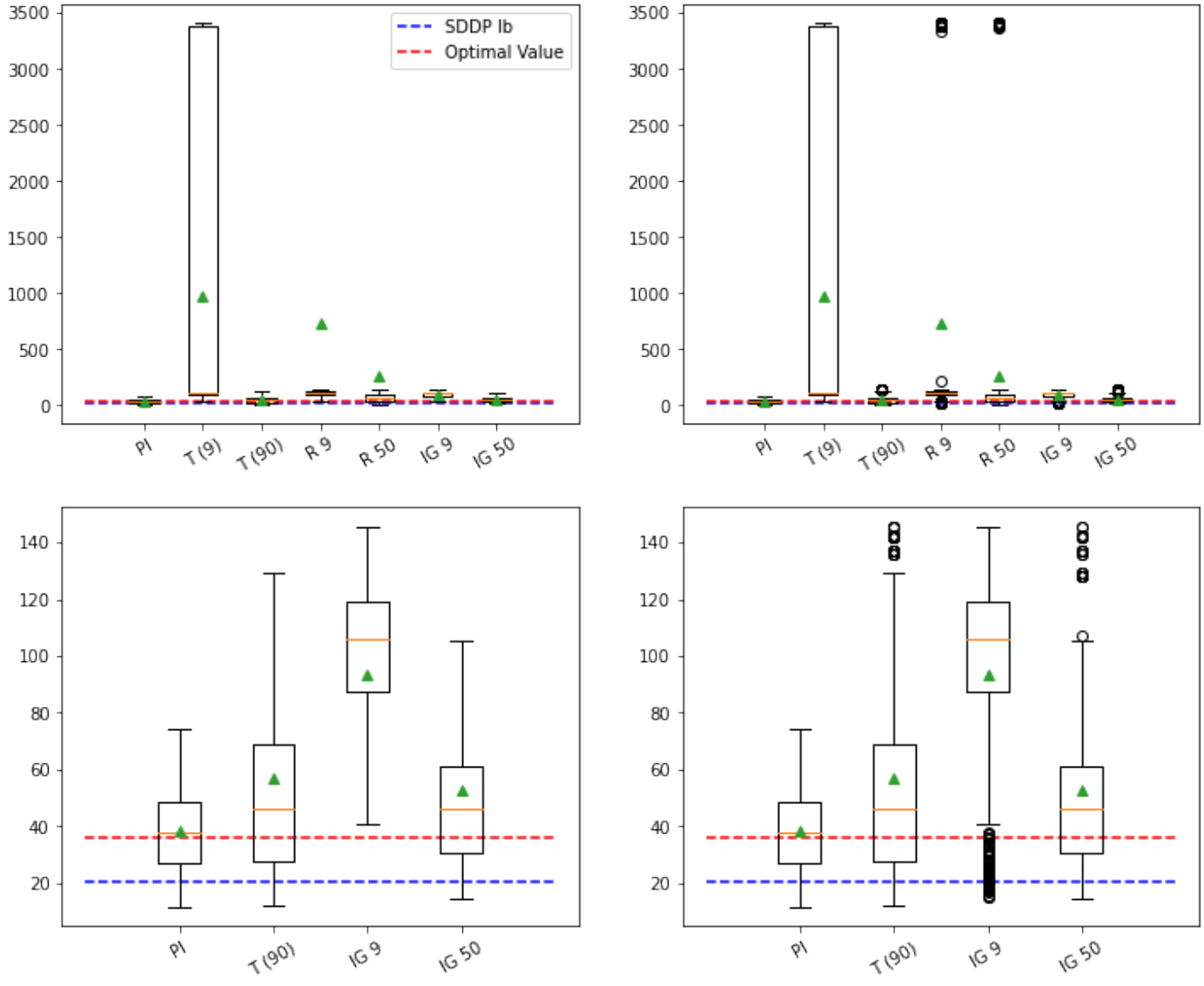
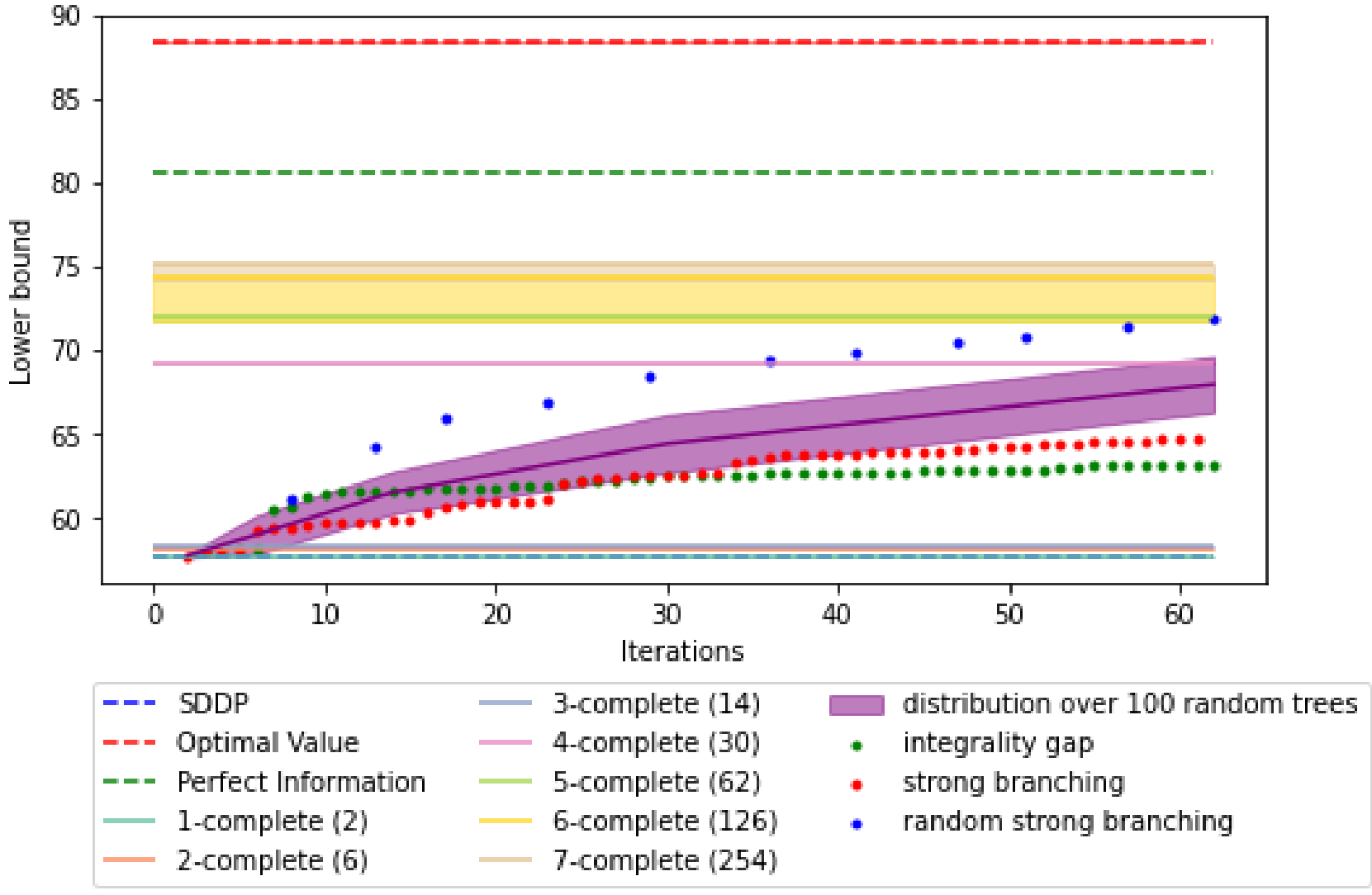


Figure: Simulations with different methods over 339 scenarios

$T = 8, Q = 2, I = 3, J = 4$


90
85
80
75
70
65
60
Lower bound
0
10
20
30
40
50
60
Iterations
SDDP
Optimal Value
Perfect Information
1-complete (2)
2-complete (6)
3-complete (14)
4-complete (30)
5-complete (62)
6-complete (126)
7-complete (254)
distribution over 100 random trees
integrality gap
strong branching
random strong branching

$T = 8, Q = 2, I = 3, J = 4$

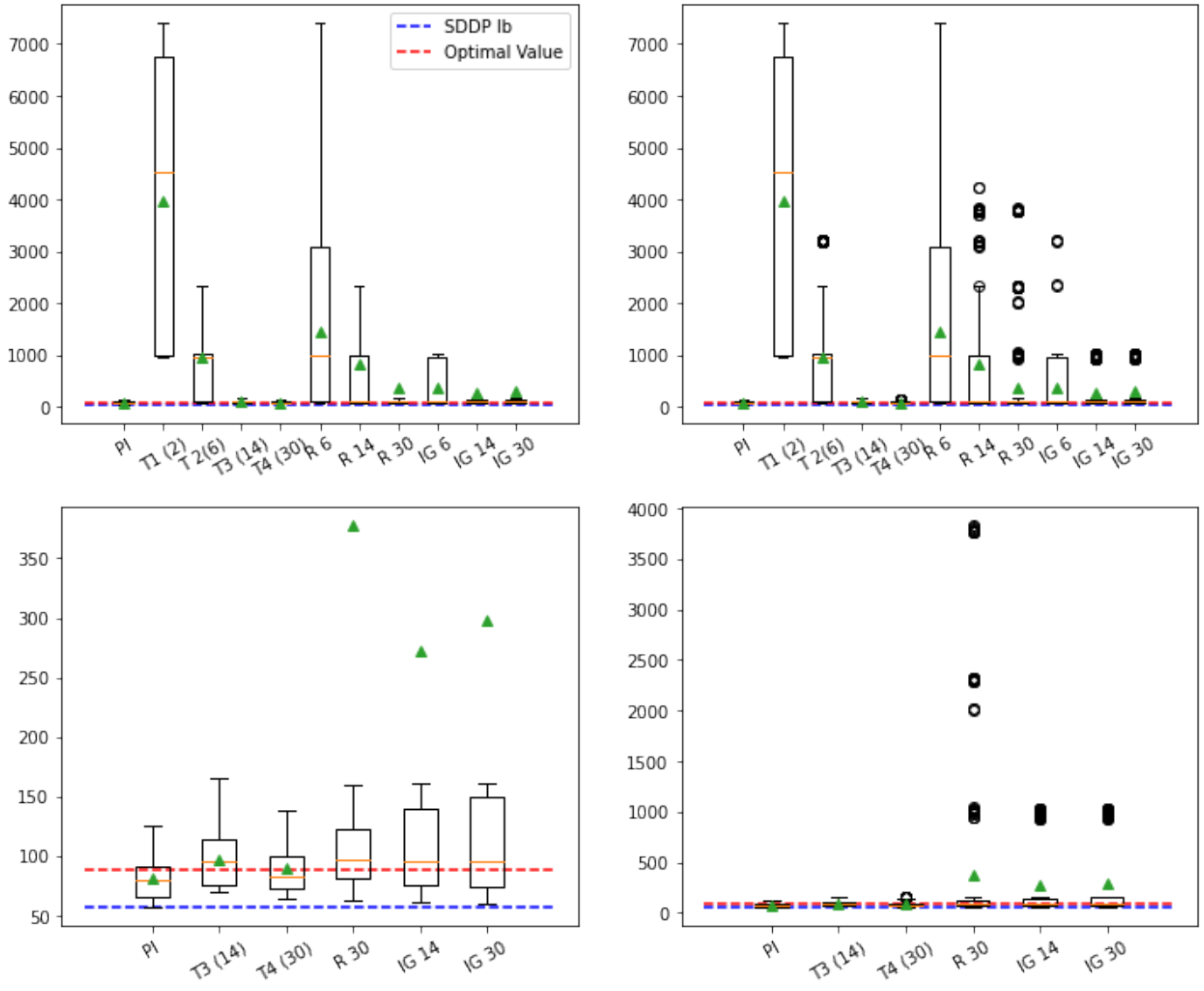


Figure: Simulations with different methods over all scenarios